Math 4550 HW due April 30, 2009

1. Let P be a 3-polytope. Prove that if none of the two dimensional faces of P is a triangle, then there is at least one vertex of degree exactly three.

2. Give a one-line proof that the graph of $\diamond^5$ is not a planar graph. What about the graph of $\diamond^4$?

3. Let Δ be an abstract simplicial complex and let F be a face of Δ. The *link* of F in Δ is the simplicial complex $\operatorname{lk} F = \{G \in \Delta : F \cap G = \emptyset, (F \cup G) \in \Delta\}$.

   Let P be a simplicial polytope and let $\Delta = \partial P$. Let F be a face of Δ with vertex set $\{v_1, \ldots, v_k\}$. Prove that as an abstract simplicial complex, the link of a vertex v in Δ equals $\partial(\operatorname{vf}(P, v))$, where $\operatorname{vf}(P, v)$ stands for the vertex figure of P at the vertex v. Prove that if F is a face of Δ with vertex set $\{v_1, \ldots, v_k\}$, then, as an abstract simplicial complex, the link of F in Δ is $\partial[\operatorname{vf}(\operatorname{vf}(\ldots \operatorname{vf}(\operatorname{vf}(P, v_1), v_2), \ldots, v_{k-1}), v_k)]$. Here, $\operatorname{vf}(\operatorname{vf}(P, v_1), v_2)$ means the vertex figure of $\operatorname{vf}(P, v_1)$ with respect to the vertex of $\operatorname{vf}(P, v_1)$ which corresponds to the edge (v_1, v_2).