Math 4550 Prelim 2 - Due May 20, 4:10 pm, 2011

You may use the following resources: Class notes (including previous homework problems) and the ‘text’. You may also use Lectures 0,2,7,8 and the parts of Lectures 3 and 4 of Ziegler that we have covered - but see the note to problem 1 below for an exception. If you have any doubt ASK. You may (are encouraged to) consult with Prof. Swartz. You may also discuss previous material (and homework problems) with Mr. Kolins. Discussion with other people or use of other resources (such as the web) is not allowed. Late exams will receive an arbitrary and unfair penalty.

1. Let $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^e$. Define $P \times Q \subseteq \mathbb{R}^{d+e}$ by

   $P \times Q \equiv \{ (x_1, \ldots, x_d, y_1, \ldots, y_e) \in \mathbb{R}^{d+e} : (x_1, \ldots, x_d) \in P \text{ and } (y_1, \ldots, y_e) \in Q. \}$

   (a) If $P$ and $Q$ are convex, then $P \times Q$ is convex.
   (b) If $P$ and $Q$ are polytopes, then $P \times Q$ is a polytope.
   (c) If $P$ and $Q$ are polytopes, then faces of $P \times Q$ are all sets of the form $F \times G$, where $F$ is a face of $P$ and $G$ is a face of $Q$. (Note: $F \times \emptyset = \emptyset \times \emptyset = \emptyset$.)

   Note: Chap. 0 of Ziegler states these as facts without proofs. Obviously you cannot just quote that!

2. Let $G$ be a connected simple graph.

   (a) Prove that if $\lambda$ does not divide $\mu_{L_G}(\hat{0}, G)$, then $G$ has a proper $\lambda$ coloring.
   (b) Prove that if $G$ has a proper 2-coloring, then $\mu_{L_G}(\hat{0}, G)$ is odd.
   (c) Show by example that there exists connected simple $G$ such that 3 divides $\mu_{L_G}(\hat{0}, G)$, but $G$ does have a proper 3-coloring.

3. Let $f = (1, f_0, f_1, f_2, 1)$ be the $f$-vector of a 3-polytope.

   (a) Prove that there exists nonnegative integers $a,b$ such that

   $f = (1, 4, 6, 4, 1) + a(0, 1, 1, 0, 0) + b(0, 0, 1, 1, 0)$.

   (b) With $a,b$ as above, prove that $b \leq 2a$ and $b = 2a$ if and only if the polytope is simplicial. Use this to show that $a \leq 2b$ and $a = 2b$ if and only if the polytope is simple.
   (c) Prove that if $a,b$ are arbitrary nonnegative integers with $b \leq 2a$ and $a \leq 2b$, then there exists a 3-polytope with $f$-vector

   $f = (1, 4, 6, 4, 1) + a(0, 1, 1, 0, 0) + b(0, 0, 1, 1, 0)$.

4. Let $P$ be a simple $d$-polytope in $\mathbb{R}^d$ and $\mathcal{O}$ a good acyclic orientation of its graph $G$. As usual, define $h^i_G$ to be the number of vertices with in-degree $i$. Let $a \in (\mathbb{R}^d)^*$ be generic for $P$. 

1
(a) Use $a$ to construct a shelling of the boundary of $P^\ast$.

(b) Suppose $\mathcal{O}$ is the good orientation associated to $a$. Prove that $h_1^\mathcal{O} = h_1(P^\ast)$.