Math 4550 Prelim 2 - Due May 13, 4:30 pm, Malott 592

You may use the following resources: Class notes (including previous homework problems), parts of the text we have covered. If you have any doubt ASK. You may (are encouraged to) consult with Prof. Swartz. Discussion with other people is not allowed. Late exams will be penalized a totally unreasonable number of points.

1. (20)
   Let $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^e$. Define $P \times Q \subseteq \mathbb{R}^{d+e}$ by

   $P \times Q \equiv \{(x_1, \ldots, x_d, y_1, \ldots, y_e) \in \mathbb{R}^{d+e} : (x_1, \ldots, x_d) \in P \text{ and } (y_1, \ldots, y_e) \in Q.\}$

   (a) If P and Q are convex, then $P \times Q$ is convex.

   (b) If P and Q are polytopes, then $P \times Q$ is a polytope.

   (c) If P and Q are polytopes, then faces of $P \times Q$ are all sets of the form $F \times G$, where F is a face of P and G is a face of Q.

2. (15) Recall that a graded (finite) poset Q with a maximum element $\hat{1}$ and minimum element $\hat{0}$ is called *Eulerian* if $\forall x \leq y,\ \mu_Q(x, y) = (-1)^{rk(y)-rk(x)}$.

   (a) If Q is Eulerian and $x \leq y$, then the interval $[x, y]$ is Eulerian.

   (b) Prove that Q is Eulerian if and only if for every interval $[x, y] \subseteq Q$, the number of elements of even rank equals the number of elements of odd rank.

3. (15) A graph G is *bipartite* if there is a partition X, Y of the vertices of G such that every edge has one vertex in X and one vertex in Y. Equivalently, G is properly 2-colorable. Prove that a connected graph G is bipartite if and only if $\mu_{L_G}(\emptyset, G)$ is odd.

4. (15) Construct a hyperplane arrangement $\mathcal{A}$ in some $\mathbb{R}^d$ all of whose sign vectors are

   $$(0,0,0,0), (0,+,+,-), (0,-,-,+), (-,0,+,-), (+,0,-,+),$$
   $$(-,-,0,-), (+,+,0,+), (-,-,-,0), (+,+,+,0),$$
   $$(-,+,+,-), (+,-,-,+), (-,-,+,-), (+,+,-,+),$$
   $$(-,-,-,-), (+,+,+,+), (-,-,-,+), (+,+,+,-).$$

5. (20) Let $f = (1, f_0, f_1, f_2, 1)$ be the f-vector of a 3-polytope.

   (a) Prove that there exists nonnegative integers a, b such that

   $$f = (1, 4, 6, 4, 1) + a(0, 1, 1, 0, 0) + b(0, 0, 1, 1, 0).$$

   (b) With a, b as above, prove that $b \leq 2a$ and $b = 2a$ if and only if the polytope is simplicial. Use this to show that $a \leq 2b$ and $a = 2b$ if and only if the polytope is simple.

(c) Prove that if a, b are arbitrary nonnegative integers with $b \leq 2a$ and $a \leq 2b$, then there exists a 3-polytope with f-vector

$$f = (1, 4, 6, 4, 1) + a(0, 1, 1, 0, 0) + b(0, 0, 1, 1, 0).$$

6. (15) Let P be a simple d-polytope in $\mathbb{R}^d$ and O a good acyclic orientation of its graph G. As usual, define h_i^O to be the number of vertices with in-degree i. Prove that $h_i^O = h_i(P^\Delta)$. You can receive full credit by proving this for acyclic orders coming from an open dense subset of $a \in (\mathbb{R}^d)^\star$.