

Math 4550 Prelim 1 - Due April 2, 4:10 pm, 2009

You may use the following resources: Class notes (including previous homework problems), parts of the text we have covered. If you have any doubt ASK. You may (are encouraged to) consult with Prof. Swartz. You may also discuss previous material (and homework problems) with Mr. Alonso. Discussion with other people is not allowed.

1. (a) Let K be a convex subset of \mathbb{R}^d which is the union of a finite number of d -simplices. Prove that K is a polytope.
- (b) Let P be a polytope in \mathbb{R}^d and let $a \in (\mathbb{R}^d)^*$. Prove that

$$\max_{x \in P} a \cdot x = \max\{a \cdot v_1, \dots, a \cdot v_n\},$$

where $\{v_1, \dots, v_n\}$ are the vertices of P .

2. (a) Let Δ_1 be the pure abstract 2-dimensional simplicial complex whose vertices are $\{1, 2, 3, 4, 5, 6, 7\}$ and whose facets are

$$\{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 6\}, \{3, 5, 6\}, \{4, 5, 7\},$$

$$\{4, 6, 7\}, \{1, 5, 6\}, \{1, 5, 7\}, \{2, 6, 7\}, \{1, 2, 6\}, \{2, 3, 7\}, \{1, 3, 7\}.$$

Is Δ_1 shellable?

- (b) Let Δ_2 be a pure abstract 2-dimensional simplicial complex whose vertices are $\{1, 2, 3, 4, 5, 6\}$ and whose facets are

$$\{1, 2, 3\}, \{1, 2, 6\}, \{1, 3, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{1, 4, 5\},$$

$$\{1, 5, 6\}, \{2, 4, 6\}, \{2, 4, 5\}, \{3, 5, 6\}.$$

Is Δ_2 shellable?

3. Let $0 < k < d$ be integers. The (k, d) -hypersimplex is the convex hull of all vectors in \mathbb{R}^d whose coordinates consist of exactly k ones and $d - k$ zeros. For example, the $(1, d)$ hypersimplex is just Δ^{d-1} since it is the convex hull of $(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)$. For another example, the $(2, 4)$ hypersimplex is the convex hull in \mathbb{R}^4 of

$$(1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 1, 0), (0, 1, 0, 1), (0, 0, 1, 1).$$

- (a) The d points which determine the $(d-1, d)$ -hypersimplex are affinely independent. Hence the $(d-1, d)$ hypersimplex is a $(d-1)$ -dimensional simplex.
- (b) A (k, d) -hypersimplex is a $d - 1$ dimensional polytope. What familiar polytope is the $(2, 4)$ hypersimplex (up to combinatorial equivalence)?
- (c) If $1 < k < d - 1$, then the (k, d) hypersimplex has at least $2d$ facets.

(d) If $1 < k < d - 1$, then the (k, d) hypersimplex has exactly $2d$ facets.

(Note: In chapter zero the text states, but gives no proof that (b) and (d) are true. Obviously you can not just quote this.)

4. (a) A 4-dimensional simplicial polytope P has 7 vertices, 21 edges and 34 triangles. How many tetrahedra does it have?
- (b) A simple 5-dimensional polytope P has 9 facets. What is the fewest number of vertices P can have?