Math 4550 HW9 - April 14, 2011

Question 1

First we show that $\pi_n$ is isomorphic to $L_{K_n}$, the poset of vertex-induced subgraphs of the complete graph on $n$ vertices. Label the vertices of $K_n$ with the elements of $[n]$. Then we define our isomorphism by mapping an element $\{A_1, \ldots, A_m\} \in \pi_n$ to the subgraph of $K_n$ consisting of all of the edges between the vertices in $A_i$ for $1 \leq i \leq m$. From its definition this map is one to one. Since $K_n$ contains an edge between each pair of vertices, any component of any element of $L_{K_n}$ must contain all of the edges between the vertices in the component. Hence our map is onto. Under our bijection, refining a partition in $\pi_n$ corresponds to removing edges in $L_{K_n}$, so our identification is order preserving in both directions and hence an isomorphism.

Note that $K_n$ has no proper $\lambda$-coloring for $\lambda < n$. Hence $1, 2, \ldots, n - 1$ are all roots of $\chi_{K_n}(\lambda)$. Also, $K_n$ has only one component, so by the discussion following Problem 48, $\chi_{K_n}(\lambda)$ is a polynomial of degree $|K_n| = n$ and contains $\lambda$ as a factor. By Problem 48 the coefficient of $\lambda^n$ in $\chi_{K_n}(\lambda)$ is $\mu(\hat{0}, \hat{0}) = 1$. Hence

$$\chi_{K_n}(\lambda) = \lambda \prod_{i=1}^{n-1} (\lambda - i).$$

Using the discussion following Problem 48, the coefficient of $\lambda^{c(K_n)} = \lambda^1$ in $\chi_{K_n}(\lambda)$ is $\mu(\hat{0}, \hat{1})$. Hence

$$\mu(\hat{0}, \hat{1}) = \prod_{i=1}^{n-1} (-i) = (-1)^{n-1}(n-1)!.$$

Question 3

First note that by Problem 53, any zonotope is centrally symmetric (up to a translation). Therefore any $n$-gon that is not centrally symmetric, in particular any $n$-gon where $n$ is odd, is not a zonotope. Since all 2-polytopes are $n$-gons for some $n$, the only polytopes that are combinatorially isomorphic to $n$-gons are other $n$-gons, so we have one direction of the result.

Now let $P$ be a centrally symmetric $2n$-gon centered at the origin. Label the vertices of $P$ cyclically as $x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}$, so that $x_{i+n} = -x_i$ for $1 \leq i \leq n$. We will show that
$P$ is a zonotope by proving that it is equal to the Minkowski sum

$$Z = \left[ -\frac{x_2 - x_1}{2}, \frac{x_2 - x_1}{2} \right] + \cdots + \left[ -\frac{x_{n+1} - x_n}{2}, \frac{x_{n+1} - x_n}{2} \right].$$

For intuition, note that the intervals defining $Z$ are parallel to a sequence of $n$ consecutive edges of $P$.

First we show that $P \subseteq Z$. Since $x_{n-1} = -x_1$ we have

$$x_1 = -\frac{x_2 - x_1}{2} - \frac{x_3 - x_2}{2} - \cdots - \frac{x_{n+1} - x_n}{2},$$

so $x_1 \in Z$. Switching the $i$th term in this sum from negative to positive adds $x_{i+1} - x_i$ to the sum. Hence switching the signs of all of the first $k$ terms for $1 \leq k \leq n$ gives the point $x_{k+1}$. By the central symmetry of $Z$, all the $x_i$ with $n+1 < i \leq 2n$ are also in $Z$. Since $Z$ is convex and contains all of the vertices of $P$, $P \subseteq Z$.

Now we show that $Z \subseteq P$. Fix any non-trivial closed half-space $H_{a,b}$ containing $P$. We must show $Z \subseteq H_{a,b}$. Since the origin is in the interior of $P$, by scaling and rotating, we can assume WLOG that $H_{a,b} = \{x = 1\}$.

We want to determine the maximum possible $x$-coordinate of any point in $Z$. To obtain this point, for each interval in the definition $Z$ we take the endpoint that has a positive $x$-coordinate (or either endpoint if their $x$-coordinates are zero) and we sum these points. Recall that the intervals defining $Z$ are parallel to a sequence of $n$ consecutive edges of $P$. Since $P$ is convex, in this sequence of $n$ consecutive edges there can be at most one sign change in the $x$-coordinates. If there are no sign changes then our sum telescopes so the maximum $x$-coordinate is given by the magnitude of the $x$-coordinate of $x_1$. If there is a sign change between the $(i-1)$st and $i$th term in the sum then again the sum telescopes and the maximum $x$-coordinate is given by the magnitude of the $x$-coordinate of $x_i$. Since $P$ is in the half-space $x \leq 1$, by central symmetry $P$ is also in the half-space $x \geq -1$. So the $x$-coordinate of all of the $x_i$ is in $[-1, 1]$, and hence the $x$-coordinate of any point in $Z$ is at most one, as desired.