Question 3

Let $P$ be a simple $d$-polytope and let $v$ be a vertex of $P$. Let $F = \psi\{\{v\}\}$; so $F$ is a facet of $P^*$. Using duality we know that $\{\{v\}, P\} \subseteq \mathcal{F}(P)$ is isomorphic to $[\emptyset, F]^* \subseteq \mathcal{F}(P)^*$.

Since $P$ is simple, $P^*$ is simplicial, so $F$ is a $(d - 1)$-simplex. We have shown that the faces of the simplex are exactly the convex hulls of the subsets of the vertices of the simplex, with containment of faces corresponding to containment of vertex sets. Therefore we have $\mathcal{F}(\Delta^{d-1}) \cong B_d$ with the isomorphism given by labeling the vertices of $\Delta^{d-1}$ as $1, \ldots, d$ and identifying a subset $S \in B_d$ with the face of $\Delta^{d-1}$ containing exactly the vertices $S$.

Further, we have $B_d \cong B_d^*$, with the isomorphism given by identifying a subset $S \in B_d$ with $[d] - S \in B_d^*$.

Hence $\{\{v\}, P\} \cong [\emptyset, F]^* \cong B_d^* \cong B_d$, as desired.