Math 4550 HW2 - Feb. 10, 2011

Question 2

\( \mathcal{V} \)-polytopes:

Let \( P = \text{ch}\{x_1, \ldots, x_n\} \) be a \( \mathcal{V} \)-polytope in \( \mathbb{R}^d \). For \( 1 \leq i \leq n \), define \( y_i = \binom{x_i}{0} \in \mathbb{R}^{d+1} \) and \( z_i = \binom{x_i}{1} \in \mathbb{R}^{d+1} \). We claim that \( \text{prism}(P) = \text{ch}\{y_1, \ldots, y_n, z_1, \ldots, z_n\} \).

\( \subseteq \) Let \( v = (v_1, \ldots, v_{d+1}) \in \text{prism}(P) \). Then \( v' := (v_1, \ldots, v_d) \in P \). Hence there exist \( a_i \geq 0 \) such that

\[ v' = \sum_{i=1}^{n} a_i x_i \quad \text{and} \quad \sum_{i=1}^{n} a_i = 1. \]

It is then straightforward to check coordinate by coordinate that we have the equality

\[ v = \sum_{i=1}^{n} a_i ((1 - v_{d+1}) y_i + v_{d+1} z_i). \]

From the definition of \( \text{prism}(P) \) we know \( 0 \leq v_{d+1} \leq 1 \). Hence all of the coefficients in the above sum are non-negative and add to one, showing that \( v \in \text{ch}\{y_1, \ldots, y_n, z_1, \ldots, z_n\} \).

\( \supseteq \) Let \( v = (v_1, \ldots, v_{d+1}) \in \text{ch}\{y_1, \ldots, y_n, z_1, \ldots, z_n\} \). Then there exist \( a_i, b_i \geq 0 \) such that

\[ v = \sum_{i=1}^{n} a_i y_i + \sum_{i=1}^{n} b_i z_i \quad \text{and} \quad \sum_{i=1}^{n} (a_i + b_i) = 1. \]

Let \( v' = (v_1, \ldots, v_d) \). Considering only the first \( d \) coordinates, the above expression for \( v \) reduces to

\[ v' = \sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} b_i x_i = \sum_{i=1}^{n} (a_i + b_i) x_i. \]

Therefore \( v' \in \text{prism}(P) \). From the definition of the \( y_i \) and \( z_i \) we know that the \( (d + 1) \)st coordinate to \( v \) is equal to \( \sum_{i=1}^{n} b_i \), which is in \([0, 1]\) from our definition of the \( a_i \) and \( b_i \). Hence \( v \in \text{prism}(P) \), as claimed.

\( \mathcal{H} \)-polytopes:

Let \( P = \cap_{i=1}^{n} H_{a_i, b_i} \) be an \( \mathcal{H} \)-polytope in \( \mathbb{R}^d \). For \( 1 \leq i \leq n \) define \( a_i = \binom{a_i'}{0} \in \mathbb{R}^{d+1} \). Note that if \( v' \in \mathbb{R}^d \) and \( v = \binom{v'}{k} \in \mathbb{R}^{d+1} \), then for any \( k \in \mathbb{R} \) we have \( v' \cdot a_i' = v \cdot a_i \). Hence \( v' \in P \) iff \( v \in \cap_{i=1}^{n} H_{a_i, b_i} \).

We claim that \( \text{prism}(P) = \cap_{i=1}^{n} H_{a_i, b_i} \cap H_{e_{d+1}, 0} \cap H_{-e_{d+1}, -1}. \)
Let \( \mathbf{v} = (v_1, \ldots, v_{d+1}) \) and let \( \mathbf{v}' = (v_1, \ldots, v_d) \). Then:

\[
\mathbf{v} \in \text{prism}(P) \iff \mathbf{v}' \in P \text{ and } 0 \leq v_{d+1} \leq 1
\]

By the definition of prism\((P)\).

\[
\iff \mathbf{v} \in \bigcap_{i=1}^n H_{a_i, b_i} \text{ and } 0 \leq v_{d+1} \leq 1
\]

By the above discussion.

\[
\iff \mathbf{v} \in \bigcap_{i=1}^n H_{a_i, b_i} \cap H_{e_{d+1}, 0} \cap H_{-e_{d+1}, -1}
\]

By the definition of \( H_{a,b} \).

To complete the proof we must show that prism\((P)\) is bounded. Since \( P \) is a polytope, the first \( d \) coordinates of any vector in prism\((P)\) are bounded. The last coordinate is in \([0,1]\) by the definition of prism\((P)\) and hence bounded.

**Question 3**

Let \( P = \text{ch}\{x_1, \ldots, x_n\} \) be a \( \mathcal{V} \)-polytope in \( \mathbb{R}^d \). Define the map \( f : \mathbb{R}^n \to \mathbb{R}^d \) by \( f(y) = A(y) \) where \( A \) is the \( d \times n \) matrix whose columns are the \( x_i \). Then \( f \) is a linear map and hence an affine map. We claim that \( f(\Delta^{n-1}) = P \).

\[ \supseteq \] Let \( \mathbf{v} \in P = \text{ch}\{x_1, \ldots, x_n\} \). Then there exist \( a_i \geq 0 \) such that

\[
\mathbf{v} = \sum_{i=1}^n a_i x_i \quad \text{and} \quad \sum_{i=1}^n a_i = 1.
\]

Let \( \mathbf{v}' = \sum_{i=1}^n a_i e_i \in \Delta^{n-1} \). Then

\[
f(\mathbf{v}') = A(\sum_{i=1}^n a_i e_i) = \sum_{i=1}^n a_i A(e_i) = \sum_{i=1}^n a_i x_i = \mathbf{v}.
\]

Hence \( \mathbf{v} \in f(\Delta^{n-1}) \).

\[ \subseteq \] Let \( \mathbf{v}' \in \Delta^{n-1} \). Then by the definition of \( \Delta^{n-1} \) we know \( \mathbf{v}' = \sum_{i=1}^n a_i e_i \) where \( a_i \geq 0 \) and \( \sum_{i=1}^n a_i = 1 \). By the above calculation, \( f(\mathbf{v}') = \sum_{i=1}^n a_i x_i \in \text{ch}\{x_1, \ldots, x_n\} = P \), as desired.

Alternatively, for \( 1 \leq i \leq n \), we have \( e_i \in \Delta^{n-1} \) and \( f(e_i) = x_i \). By Question 1.5 we know that the affine image of a convex set is convex. Therefore, \( f(\Delta^{n-1}) \) is a convex set containing all of the \( x_i \). Since \( P = \text{ch}\{x_1, \ldots, x_n\} \) is the smallest convex set containing all of the \( x_i \) we have the desired containment.