Math 4320 Prelim 2 - Due May 1, 2:15 pm, 2013

You may use the following resources: Class notes (including previous homework problems), parts of the text we have covered. These include Chapter 2, sections 1-7, Chapter 3, sections 1-4, parts of section 5, sections 7 and 8. If you have any doubt ASK. You may (are encouraged to) consult with Prof. Swartz. You may also discuss previous material (and homework problems) with Mr. Einstein. Discussion with other people or use of other resources (such as the web) is not allowed. Late exams will receive an arbitrary and unfair penalty.

1. Let \( K = \mathbb{F}_5[x]/(x^3 + x^2 + 2) \).
   (a) Prove that \( K \) is a field with 125 elements.
   (b) We have seen that every element of \( K \) is a coset of the form
   \[ [ax^2 + bx + c], \]
   with \( a, b, c \in \mathbb{F}_5 \). Write \( [x + 1]^{251} \) in this form. All computations should be done by hand. No electronic calculating devices of any kind. Show all work.

2. A field is \( E \) is algebraically closed if every polynomial \( f(x) \in E[x] \) has a root in \( E \).
   (a) Prove that if \( E \) is algebraically closed, then every polynomial \( f(x) \in E[x] \) splits in \( E \).
   (b) Prove or disprove: Every algebraically closed field has infinitely many elements.

3. Let \( (R, +, \cdot) \) be a commutative ring with multiplicative identity \( 1_R \). Define new binary operations \( \oplus \) and \( \otimes \) on \( R \) by
   \[ a \oplus b = a + b + 1_R, \quad a \otimes b = a + b + (a \cdot b). \]
   (a) Prove that \( (R, \oplus, \otimes) \) is a commutative ring.
   (b) Prove that as rings, \( (R, +, \cdot) \) and \( (R, \oplus, \otimes) \) are isomorphic.

4. A nonempty subset \( S \) of a commutative ring \( R \) is multiplicatively closed if whenever \( s, t \in S \), then \( st \in S \). For instance, if \( R \) is an integral domain, then \( R - \{0\} \) is multiplicatively closed. For the remainder of this problem \( S \) is a fixed multiplicatively closed subset of a commutative ring \( R \). Define a binary relation \( \sim \) on \( R \times S \) by \( (a, s) \sim (b, t) \) if there exists a \( u \in S \) such that \( u(at - bs) = 0 \).
   (a) Prove that \( \sim \) is an equivalence relation. For the rest of the problem denote the equivalence classes of \( R \times S/ \sim \) by \( R_S \).
   (b) For \([a, s] \) and \([b, t] \) in \( R_S \) define \([a, s] \otimes [b, t] \) by \([ab, st] \). Prove that \( \otimes \) is a well-defined binary operation on \( R_S \), is associative, and has an identity.
(c) If we define \([a, s] \oplus [b, t] = [at + bs, st]\), then \((R, \oplus, \otimes)\) is a commutative ring. (Do not prove this!) Let \(R = \mathbb{Z}/6\mathbb{Z}\) and \(S = \{2, 4\}\). What familiar ring is \(RS\)?