Problem 3.4.1 A lot of people had trouble with the last question, so please read the solution.

Solution:

\[
\begin{bmatrix}
2 & 4 & 6 & 4 & b_1 \\
2 & 5 & 7 & 6 & b_2 \\
2 & 3 & 5 & 2 & b_3 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
2 & 4 & 6 & 4 & b_1 \\
0 & 1 & 1 & 2 & b_2 - b_1 \\
0 & -1 & -1 & -2 & b_3 - b_1 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
2 & 4 & 6 & 4 & b_1 \\
0 & 1 & 1 & 2 & b_2 - b_1 \\
0 & 0 & 0 & 0 & b_3 + b_2 - 2b_1 \\
\end{bmatrix}
\]

Hence, \( Ax = b \) has a solution if \( b_3 + b_2 - 2b_1 = 0 \). Also, we see that the pivot columns of \( A \) are its first and second columns. Hence, the column space is the set of all combinations of \((2, 2, 2)\) and \((4, 5, 3)\). This is, of course, precisely the plane \( b_3 + b_2 - 2b_1 = 0 \).

The nullspace contains all combinations of \( s_1 = (-1, -1, 1, 0) \) and \( s_2 = (2, -2, 0, 1) \).

Also,

\[
[R \ d] = \begin{bmatrix}
1 & 0 & 1 & -2 & 4 \\
0 & 1 & 1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

gives the particular solution \( x_p = (4, -1, 0, 0) \). Finally, the complete solution is

\[
x_c = \begin{bmatrix}
4 \\
-1 \\
0 \\
0 \\
\end{bmatrix}
+ c_1 \begin{bmatrix}
-1 \\
-1 \\
1 \\
0 \\
\end{bmatrix}
+ c_2 \begin{bmatrix}
2 \\
-2 \\
0 \\
1 \\
\end{bmatrix}
\text{ for } c_1, c_2 \in \mathbb{R}.
\]

Problem 3.4.7

Solution:

\[
\begin{bmatrix}
1 & 2 & -2 & b_1 \\
2 & 5 & -4 & b_2 \\
4 & 9 & -8 & b_3 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & -2 & b_1 \\
0 & 1 & 0 & b_2 - 2b_1 \\
0 & 0 & 0 & b_3 - 2b_1 - b_2 \\
\end{bmatrix}
\]

Hence, the system is solvable if \( b_3 - 2b_1 - b_2 = 0 \). Back-substitution gives the particular solution to \( Ax = b \) and the special solution to \( Ax = 0 \):

\[
x = \begin{bmatrix}
5b_1 - 2b_2 \\
b_2 - 2b_1 \\
0 \\
\end{bmatrix}
+ x_3 \begin{bmatrix}
2 \\
0 \\
1 \\
\end{bmatrix}.
\]
Problem 3.4.7

Solution:

\[
\begin{bmatrix}
1 & 3 & 1 & b_1 \\
3 & 8 & 1 & b_2 \\
2 & 4 & 0 & b_3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 3 & 1 & b_1 \\
0 & -1 & -1 & b_2 - 2b_1 \\
0 & -2 & -2 & b_3 + b_2 - 5b_1
\end{bmatrix}.
\]

One more step gives \([0, 0, 0, 0] = \text{row3} - 2(\text{row2}) + 4(\text{row1})\) provided that \(b_3 - 2b_2 + 4b_1 = 0\).

Problem 3.4.32

Solution:

For special solutions \((2, 2, 1, 0)\) and \((3, 1, 0, 1)\) with free variables \(x_3, x_4\), we get

\[R = \begin{bmatrix}
1 & 0 & -2 & -3 \\
0 & 1 & -2 & -1
\end{bmatrix}.
\]

\(A\) can be any invertible 2 by 2 matrix times this \(R\), e.g., if the 2 by 2 matrix is the identity matrix, we get \(A = R\).

Problem 3.4.32

Solution:

\[
A = \begin{bmatrix}
1 & 3 & 1 \\
1 & 2 & 3 \\
2 & 4 & 6 \\
1 & 1 & 5
\end{bmatrix}
\]

factors into \(LU = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
2 & 2 & 1 & 0 \\
1 & 2 & 0 & 1
\end{bmatrix}\begin{bmatrix}
1 & 3 & 1 \\
0 & -1 & 2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

The rank is, therefore, 2. The special solution to \(Ax = 0\) and \(Ux = 0\) is \(s = (-7, 2, 1)\).

For \(b = (1, 3, 6, 5)\), since it is the last column of \(A\), we get the particular solution \((0, 0, 1)\) to \(Ax = b\). Hence the complete solution is \(x = (0, 0, 1) + c(-7, 2, 1)\), for \(c \in \mathbb{R}\).

For \(b = (1, 0, 0, 0)\) elimination leads to \(Ux = (1, -1, 0, 1)\) and the equation on the fourth component becomes \(0 = 1\). Hence, there is no solution for this \(b\). If you had trouble solving this problem, you are most likely confused about the definition of this factorization. Go back and read the book!
Problem 8.4.2
Please read this solution, since, even though it is supposedly in the back of the book, the book is wrong. I guess a lot of people read this alleged solution and copied it, since a lot of people got that result. However, that result is clearly wrong, since the point (6,0) does not satisfy the condition $2x+y \leq 6$, and the point (0,6) does not satisfy the condition $x+2y \leq 6$.

Solution:
The two conditions give you the four edges (I took edges to mean vertices here) (0,0), (2,2), (3,0) and (0,3). Hence, the cost $c = 2x - y$ is hence minimized at $(x,y) = (0,3)$, for which we get $c = -3$.

Problem 3.5.16
Solution:
A warning: none of these bases are unique.

- $(1,1,1,1)$ for the space of all constant vectors $(c,c,c,c)$ for $c \in \mathbb{R}$.
- $(1,-1,0,0),(1,0,-1,0),(1,0,0,-1)$ for the space of vectors with sum of components $= 0$.
- $(1,-1,-1,0),(1,-1,0,-1)$ for the space perpendicular to $(1,1,0,0)$ and $(1,0,1,1)$.
- The columns of the identity matrix $I$ are a basis for its column space, the empty set is a basis (by convention) for $\mathbf{N}(I) = \{0\}$.

Problem 3.5.21
Solution:

- The only solution to $Ax = 0$ is $x = 0$ because the columns of $A$ are linearly independent.
- $Ax = b$ is solvable because the columns span $\mathbb{R}^5$. A key point to notice is that a basis gives exactly one solution for every $b$. 