You can find here the solutions to the problems whose solution is not on the back of the book.

**Problem 2.5.2**

**Solution:**
A row exchange gives $P^2 = I$, and hence $P^{-1} = P$. Hence $P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Actually, always $P^{-1} = P^T$, as we have seen in Section 2.7.

**Problem 2.5.11**

**Solution:** A lot of people did not read the instructions carefully and instead gave examples of invertible matrices whose sum is invertible, or singular matrices whose sum is singular. Be careful! Reading instruction carefully should be the first step in these problems... and in the prelim!

1. Let $A$ be an invertible matrix. Then if $B = -A$ then certainly $A + B = 0$ is not invertible.

2. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are both singular, but $A + B = I$ is invertible.

**Problem 2.5.23**

**Solution:**
This problem was almost exclusively computational. However, some people made mistakes. You should practice inverting matrices for the prelim!

$$[A I] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{0}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{3}{4} \\ \frac{1}{3} & -\frac{2}{3} & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & \frac{4}{3} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & \frac{3}{2} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{2}{3} & -\frac{3}{4} \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & \frac{4}{3} & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -1 & \frac{1}{2} \\ -\frac{3}{4} & \frac{3}{2} & -\frac{3}{4} \\ \frac{1}{3} & -\frac{2}{3} & 1 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{3}{4} & \frac{3}{2} & -\frac{3}{4} \\ \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{bmatrix} = [I A^{-1}].$$
Hence,

\[ A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{bmatrix} \].

Problem 2.5.29

Solution:

1. True (If \( A \) has a row of zeros, then every \( AB \) has too, and \( AB = I \) is impossible).

2. False (the matrix of all ones is singular even with diagonal 1’s).

3. True (the inverse of \( A^{-1} \) is \( A \) and the inverse of \( A^2 \) is \( (A^{-1})^2 \)).

Problem 2.5.32

Solution:

\[ A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \].

When the triangular \( A \) alternates 1 and -1 on its diagonal, \( A^{-1} \) is bidiagonal with 1’s on the diagonal and first superdiagonal.

Problem 2.6.16

Solution:

\[ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} c = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \text{ gives } c = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}. \] Then, \[ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \text{ gives } x = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}. \] Those are the forward and back substitution steps for

\[ Ax = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}. \]

Hence, we get

\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}. \]
Problem 2.6.17

1. $L$ goes to $I$.
2. $I$ goes to $I^{-1}$.
3. $LU$ goes to $U$. Elimination by $L^{-1}!$

Problem 2.7.19

1. The transpose of $R^T AR$ is $R^T A^T R^{TT} = R^T AR$. This is $n$ by $n$ when $A^T = A$ (any $m$ by $n$ matrix $R$).
2. $(R^T R)_{jj} = (\text{column } j \text{ of } R) \cdot (\text{column } j \text{ of } R) = (\text{length square of column } j) \geq 0$.

Problem 2.7.40

Solution:
Start from $Q^T Q = I$, as in $[\begin{array}{c} q_1^T \\ q_2^T \end{array}] [\begin{array}{cc} q_1 & q_2 \end{array}] = [\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}]$. 

1. The diagonal entries give $q_1^T q_2 = 1$ and $q_2^T q_2 = 1$. Hence, these are unit vectors.
2. The off-diagonal entry is $q_i^T q_2 = 0$ (and in general $q_i^T q_j = 0$ for $i \neq j$).
3. The leading example for $Q$ is the rotation matrix $[\begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array}]$. 