Gorenstein rings through face rings of manifolds.

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May 8, 2008

Abstract

The face ring of a homology manifold (without boundary) modulo a generic system of parameters is studied. Its socle is computed and it is verified that a particular quotient of this ring is Gorenstein. This fact is used to prove that the sphere $g$-conjecture implies all enumerative consequences of its far reaching generalization (due to Kalai) to manifolds. A special case of Kalai’s manifold $g$-conjecture is established for homology manifolds that have a codimension-two face whose link contains many vertices.

1 Introduction

In 1980 Stanley proved the necessity of McMullen’s conjectured description of $f$-vectors of boundaries of simplicial convex polytopes [10]. At about the same time, Billera and Lee demonstrated that McMullen’s conditions were sufficient [1]. Since then, one of the most central problems in the field of face numbers of simplicial complexes is the $g$-conjecture. In its most optimistic form it states that, just as in the case of polytope boundaries, the face ring of a homology sphere modulo a generic system of parameters has a Lefschetz element. In the middle 90’s Kalai suggested a far reaching generalization of this conjecture to all homology manifolds [6, Section 7]. It is a remarkable fact that all of the enumerative consequences of Kalai’s conjecture are implied by the apparently weaker $g$-conjecture. This follows from our main result, Theorem 1.4, that a particular quotient of the face ring

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$^*$Research partially supported by Alfred P. Sloan Research Fellowship and NSF grant DMS-0500748
$^\dagger$Research partially supported by NSF grant DMS-0600502
of a homology manifold is Gorenstein. We also verify a special case of Kalai’s conjecture when the complex has a codimension-two face whose link contains many vertices.

The main objects we will consider are Buchsbaum complexes and more specifically homology manifolds. Historically, Buchsbaum complexes were defined algebraically. Here we adopt the following theorem of Schenzel [9] as our definition. Let $k$ be an infinite field of an arbitrary characteristic, and let $\tilde{H}_i(\Delta)$ be the $i$-th reduced simplicial homology of $\Delta$ with coefficients in $k$.

**Definition 1.1** A $(d - 1)$-dimensional simplicial complex $\Delta$ is called Buchsbaum (over $k$) if it is pure and for every non-empty face $\tau \in \Delta$, $\tilde{H}_i(\text{lk}_\tau) = 0$ for all $i < d - |\tau| - 1$, where $\text{lk}_\tau = \{\sigma \in \Delta : \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta\}$ is the link of $\tau$ in $\Delta$.

We say that $\Delta$ is a homology manifold (without boundary) if it is Buchsbaum and in addition $\tilde{H}_{d-|\tau|-1}(\text{lk}_\tau) \cong k$ for all $\emptyset \neq \tau \in \Delta$. A $k$-homology sphere is a complex $\Delta$ such that for all $\tau \in \Delta$, including $\tau = \emptyset$,

$$
\tilde{H}_i(\text{lk}_\tau) = \begin{cases} 
0 & \text{if } i < d - |\tau| - 1, \\
k & \text{if } i = d - |\tau| - 1.
\end{cases}
$$

In particular, a $k$-homology sphere is a $k$-homology manifold, and a triangulation of a topological sphere (topological manifold, resp.) is a $k$-homology sphere (k-homology manifold, resp.) for any field $k$.

If $\Delta$ is a simplicial complex on $[n]$, then its face ring (or the Stanley-Reisner ring) is $k[\Delta] := k[x_1, \ldots, x_n]/I_\Delta$, where $I_\Delta = (x_{i_1}x_{i_2}\cdots x_{i_k} : \{i_1 < i_2 < \cdots < i_k \} \notin \Delta)$.

Various combinatorial and topological invariants of $\Delta$ are encoded in the algebraic invariants of $k[\Delta]$ and vice versa. For instance, if $\Delta$ is $(d - 1)$-dimensional complex, then the Krull dimension of $k[\Delta]$, $\dim k[\Delta]$, is equal to $d$. In this case, a set of $d$ linear forms $\theta_1, \ldots, \theta_d \in k[\Delta]$ is called a linear system of parameters (abbreviated, l.s.o.p.) if $k(\Delta) := k[\Delta]/(\theta_1, \ldots, \theta_d)$ has Krull dimension zero (equivalently, $k(\Delta)$ is a finite-dimensional $k$-space). Assuming $k$ is infinite, an l.s.o.p. always exists: a generic choice of $\theta_1, \ldots, \theta_d$ does the job.

One invariant that measures how far $\Delta$ is from being a homology sphere is the socle of $k(\Delta)$, $\text{Soc} k(\Delta)$, where for a $k[x_1, \ldots, x_n]$- or $k[\Delta]$-module $M$,

$$\text{Soc} M := \{y \in M : x_i \cdot y = 0 \text{ for all } i = 1, \ldots, n\}.$$ 

When $\Delta$ is a homology sphere, $\text{Soc} k(\Delta)$ is a 1-dimensional $k$-space. Since $k[\Delta]$ is a graded $k$-algebra for any $\Delta$, the ring $k(\Delta)$ and its ideal $\text{Soc} k(\Delta)$ are graded as well. We denote by $k(\Delta)_i$ and $(\text{Soc} k(\Delta))_i$ their $i$-th homogeneous components. It is well known (for instance, see [11, Lemma III.2.4(b)]) that for any $(d - 1)$-dimensional $\Delta$, $k(\Delta)_i$, and hence also $(\text{Soc} k(\Delta))_i$, vanish for all $i > d$. If $\Delta$ is a Buchsbaum complex, then for $i < d$, $(\text{Soc} (k(\Delta)))_i$ can be expressed in terms of the local cohomology modules of $k[\Delta]$ with respect to the irrelevant ideal, $H^i(k[\Delta])$, as follows, see [7, Theorem 2.2].
Theorem 1.2 Let $\Delta$ be a $(d - 1)$-dimensional simplicial complex. If $\Delta$ is Buchsbaum, then for all $0 \leq i \leq d$,

$$(\text{Soc } k(\Delta))_i \cong \left( \bigoplus_{j=0}^{d-1} \binom{d}{j} H^j(k[\Delta])_{i-j} \right) \bigoplus S_{i-d},$$

where $S$ is a graded submodule of $\text{Soc } H^d(k[\Delta])$ and $rM$ denotes the direct sum of $r$ copies of $M$.

While Theorem 1.2 identifies a big chunk of the socle of $k(\Delta)$, its other part, $S$, remains a mystery. Here we solve this mystery in the special case of a connected orientable $k$-homology manifold without boundary, thus verifying Conjecture 7.2 of [7]. A connected $k$-homology manifold $\Delta$ without boundary is called orientable if $\check{H}_{d-1}(\Delta) \cong k$.

Theorem 1.3 Let $\Delta$ be a $(d - 1)$-dimensional connected orientable $k$-homology manifold without boundary. Then $\dim_k S = \dim_k S_0 = 1$. In particular, $\dim \text{Soc } k(\Delta)_i = \binom{d}{i} \beta_{i-1}$, where $\beta_{i-1} := \dim_k \check{H}_{i-1}(\Delta)$ is the $i$-th reduced Betti number of $\Delta$.

A graded $k$-algebra of Krull dimension zero is called Gorenstein if its socle is a 1-dimensional $k$-space (see [11, p. 50] for many other equivalent definitions). Let

$I := \bigoplus_{i=0}^{d-1} \text{Soc } k(\Delta)_i$ and $\overline{k(\Delta)} := k(\Delta)/I$.

(Note that the top-dimensional component of the socle is not a part of $I$). If $\Delta$ is a homology sphere, then $I = 0$ and $\overline{k(\Delta)} = k(\Delta)$ is a Gorenstein ring [11, Theorem II.5.1]. What if $\Delta$ is a homology manifold other than a sphere? How far is $k(\Delta)$ from being Gorenstein in this case? The answer (that was conjectured in [7, Conjecture 7.3]) turns out to be surprisingly simple:

Theorem 1.4 Let $\Delta$ be a $(d - 1)$-dimensional connected simplicial complex. If $\Delta$ is an orientable $k$-homology manifold without boundary then $\overline{k(\Delta)}$ is Gorenstein.

In [9], Schenzel computed the Hilbert function of $k(\Delta)$ for a Buchsbaum complex $\Delta$ in terms of its face and Betti numbers. It follows from Theorem 1.3 combined with Schenzel’s formula and Dehn-Sommerville relations [3] that for a connected orientable homology manifold $\Delta$, the Hilbert function of $\overline{k(\Delta)}$ is symmetric, that is,

$$\dim_k \overline{k(\Delta)}_i = \dim_k \overline{k(\Delta)}_{d-i} \quad \text{for all } 0 \leq i \leq d. \quad (1)$$

As the Hilbert function of a Gorenstein ring of Krull dimension zero is always symmetric [11, p. 50], Theorem 1.4 gives an alternative algebraic proof of (1).

Theorems 1.3 and 1.4 are ultimately related to the celebrated $g$-conjecture that provides a complete characterization of possible face numbers of homology spheres. The most
optimistic version of this conjecture is a very strong manifestation of the symmetry of the Hilbert function. It asserts that if $\Delta$ is a $(d-1)$-dimensional homology sphere, and $\omega$ and $\Theta = \{\theta_1, \ldots, \theta_d\}$ are sufficiently generic linear forms, then for $i \leq d/2$, multiplication
\[ \omega^{d-2i} : k(\Delta)_i \longrightarrow k(\Delta)_{d-i} \]
is an isomorphism. At present this conjecture is known to hold only for the class of polytopal spheres [10] and edge decomposable spheres [4], [5].

Kalai’s far-reaching generalization of the $g$-conjecture [6] posits that if $\Delta$ is an orientable homology manifold and $\omega, \Theta$ are sufficiently generic, then
\[ \omega^{d-2i} : k(\Delta)_i \longrightarrow k(\Delta)_{d-i} \]
is still an isomorphism for all $i \leq d/2$. Let $h''_i = \dim_k k(\Delta)_i$. Given a system of parameters and $\omega$ which satisfy Kalai’s conjecture it is immediate that multiplication $\omega : k(\Delta)_i \rightarrow k(\Delta)_{i+1}$ is an injection for $i < d/2$. So, $h''_0 \leq \cdots \leq h''_{\lfloor d/2 \rfloor}$, and examination of $k(\Delta)/(\omega)$ shows that the nonnegative integer vector $(h''_0, h''_1 - h''_0, \ldots, h''_{\lfloor d/2 \rfloor} - h''_{\lfloor d/2 \rfloor-1})$ is an M-vector, i.e. satisfies Macaulay’s nonlinear arithmetic conditions (see [11, p. 56]) for the Hilbert series of a homogeneous quotient of a polynomial ring. In fact, as we will see in Theorem 3.2, these two conclusions follow from the $g$-conjecture.

For Kalai’s conjecture we prove this special case.

**Theorem 1.5** Let $\Delta$ be a $(d-1)$-dimensional orientable $k$-homology manifold with $d \geq 3$. If $\Delta$ has a $(d-3)$-dimensional face $\tau$ whose link contains all of the vertices of $\Delta$ that are not in $\tau$, then for generic choices of $\omega$ and $\Theta$, $\omega^{d-2} : k(\Delta)_1 \longrightarrow k(\Delta)_{d-1}$ is an isomorphism.

The condition that the link of $\tau$ contains all of the vertices of $\Delta$ that are not in $\tau$ is equivalent to saying that every vertex of $\Delta$ is in the star of $\tau$, st $\tau := \{\sigma \in \Delta : \sigma \cup \tau \in \Delta\}$. This condition is not as restrictive as one might think: the results of [12, Section 5] imply that every connected homology manifold $M$ without boundary that has a triangulation, always has a triangulation $\Delta$ satisfying this condition.

The outline of the paper is as follows. We verify Theorem 1.3 in Section 2. The main ingredient in the proof is Gräbe’s explicit description of $H^d(k[\Delta])$ as a $k[\Delta]$-module in terms of the simplicial (co)homology of the links of faces of $\Delta$ and maps between them [2]. Theorem 1.4 is proved in Section 3 and is used to explore the relationship between the $g$-conjecture and Kalai’s conjecture. Section 4 is devoted to the proof of Theorem 1.5. The proofs of Theorems 1.4 and 1.5 rely heavily on a result from [12] relating $k(\text{lk} v)$ (for a vertex $v$ of $\Delta$) to the principal ideal $(x_v) \subset k(\Delta)$.

# 2 Socles of homology manifolds

The goal of this section is to verify Theorem 1.3. To do so we analyze $\text{Soc} H^d(k[\Delta])$ and prove the following.
If $\Delta$ is a connected orientable $(d-1)$-dimensional $k$-homology manifold without boundary, then $\text{Soc } H^d(k[\Delta])_i = 0$ for all $i \neq 0$.

The proof relies on results from [2]. We denote by $|\Delta|$ the geometric realization of $\Delta$. For a face $\tau \in \Delta$, let $\text{cost } \tau := \{ \sigma \in \Delta : \sigma \nsubseteq \tau \}$ be the contrastar of $\tau$, let $H^i(\Delta, \text{cost } \tau)$ be the simplicial $i$-th cohomology of a pair (with coefficients in $k$), and for $\tau \subset \sigma \in \Delta$, let $\iota^* \in H^i(\Delta, \text{cost } \sigma)$ be the map $H^i(\Delta, \text{cost } \sigma) \to H^i(\Delta, \text{cost } \tau)$ induced by inclusion $\iota : \text{cost } \tau \to \text{cost } \sigma$. Also, if $\emptyset \neq \tau \in \Delta$, let $\hat{\tau}$ be the barycenter of $\tau$. Finally, for a vector $U = (u_1, \ldots, u_n) \in \mathbb{Z}^n$, let $s(U) := \{ l : u_l \neq 0 \} \subseteq [n]$ be the support of $U$, let $\{ e_i \}_{i=1}^n$ be the standard basis for $\mathbb{Z}^n$, and let $\mathbb{N}$ denote the set of nonnegative integers.

**Theorem 2.2** [Gräbe] The following is an isomorphism of $\mathbb{Z}^n$-graded $k[\Delta]$-modules

$$H^{i+1}(k[\Delta]) \cong \bigoplus_{s(U) \in \Delta} H^i(\Delta, \text{cost } s(U)),$$

where the $k[\Delta]$-structure on the $U$-th component of the right-hand side is given by

$$x_i = \begin{cases} 
\text{0-map,} & \text{if } l \notin s(U) \\
\text{identity map,} & \text{if } l \in s(U) \text{ and } l \in s(U + e_l) \\
\iota^* : H^i(\Delta, \text{cost } s(U)) \to H^i(\Delta, \text{cost } s(U + e_l)), & \text{otherwise.}
\end{cases}$$

We remark that the isomorphism of (2) on the level of vector spaces (rather than $k[\Delta]$-modules) is due to Hochster, see [11, Section II.4].

**Proof of Theorem 2.1:** In view of Theorem 2.2, to prove that $\text{Soc } H^d(k[\Delta])_i = 0$ for all $i \neq 0$, it is enough to show that for every $\emptyset \neq \tau \in \Delta$ and $l \in \tau$, the map $\iota^* : H^{d-1}(\Delta, \text{cost } \tau) \to H^{d-1}(\Delta, \text{cost } \sigma)$, where $\sigma = \tau - \{ l \}$, is an isomorphism. Assume first that $\sigma \neq \emptyset$. Consider the following diagram.

\[
\begin{array}{ccc}
H^{d-1}(|\Delta|) & \xrightarrow{(j^*)^{-1}} & H^{d-1}(|\Delta|, |\Delta| - \hat{\tau}) \\
\downarrow & & \downarrow \iota^* \\
H^{d-1}(|\Delta|) & \xrightarrow{(j^*)^{-1}} & H^{d-1}(|\Delta|, |\Delta| - \hat{\tau}) \\
\end{array}
\]

The two $f^*$ maps are induced by inclusion and are isomorphisms by the usual deformation retractions. $j^*$ is also induced by inclusion. Since $\Delta$ is connected and orientable, all of the spaces are one-dimensional and $j^*$ is an isomorphism, so that $(j^*)^{-1}$ is well-defined and is an isomorphism as well. Hence compositions $f^* \circ (j^*)^{-1}$ are isomorphisms. The naturality of $j^*$ implies that the diagram is commutative. It follows that $\iota^*$ is an isomorphism. If $\sigma = \emptyset$, replace in the above diagram $H^{d-1}(|\Delta|, |\Delta| - \hat{\tau})$ with $H^{d-1}(|\Delta|)$. The same reasoning applies.

We are now in a position to complete the proof of Theorem 1.3.
Proof of Theorem 1.3: Since $S_i$ is a subspace of $\text{Soc} H^d(k[\Delta])_i$ (Theorem 1.2), and since the latter space is the zero-space whenever $i \neq 0$ (see Theorem 2.1), it follows that $S_i = 0$ for all $i \neq 0$. For $i = 0$, we have

$$\dim S_0 = \dim \text{Soc} k(\Delta)_d = \dim k(\Delta)_d = \beta_{d-1}(\Delta) = 1.$$  

Here the first step is by Theorem 1.2, the second step is a consequence of $k(\Delta)_d$ being the last non vanishing homogeneous component of $k(\Delta)$, and the third step is by Schenzel’s formula [9] (see also [11, Theorem II.8.2]). The “In particular”-part then follows from Theorem 1.2, isomorphism (2), and the standard fact that $H^i−1(\Delta, \text{cost } \tau) \cong \tilde{H}_{i−|\tau|−1}(lk \tau)$, see e.g. [2, Lemma 1.3]. □

3 Gorenstein property

In this section we verify Theorem 1.4 and use it to discuss connections between various g-conjectures. To prove that $k(\Delta) = k(\Delta)/I$ is Gorenstein, where $\Delta$ is a $(d−1)$-dimensional connected orientable homology manifold and $I = \oplus_{j=0}^{d−1} \text{Soc} k(\Delta)_j$, we have to check that the operation of moding out by $I$ does not introduce new socle elements. This turns out to be a simple application of [12, Proposition 4.24], which we review now.

Let $\Theta = \{\theta_1, \ldots, \theta_d\}$ be an l.s.o.p. for $k[\Delta]$ and let $v$ be a vertex of $\Delta$. Fix a facet $\tau = \{v = v_1, v_2, \ldots, v_d\}$ that contains $v$. By doing Gaussian elimination on the $d \times n$-matrix whose $(i, j)$-th entry is the coefficient of $x_j$ in $\theta_i$, we can assume without loss of generality that $\theta_i = x_v + \sum_{j \notin \tau} \theta_{i,j} x_j$. Denote by $\theta'_i$ the linear form obtained from $\theta_i$ by removing all summands involving $x_j$ for $\{j\} \notin \text{lk } v$. Then $\Theta' := \{\theta'_2, \ldots, \theta'_d\}$ can be considered as a subset of $k[\text{lk } v]$. Moreover, it is easy to check, say, using [11, Lemma III.2.4(a)], that $\Theta'$ forms an l.s.o.p. for $k[\text{lk } v]$. The ring $k[\text{lk } v] := k[\text{lk } v]/(\Theta')$ has a natural $k[x_1, \ldots, x_v, \ldots, x_n]$-module structure (if $j \neq v$ is not in the link of $v$, then multiplication by $x_j$ is the zero map), and defining

$$x_v \cdot y := -\theta'_1 \cdot y \quad \text{for } y \in k[\text{lk } v]$$

extends it to a $k[x_1, \ldots, x_n]$-module structure. Proposition 4.24 of [12] asserts the following.

Theorem 3.1 Let $\Delta$ be an orientable homology manifold. The map

$$\phi : k[\text{lk } v]/(\Theta') \to (x_v) (k[\Delta]/(\Theta)) \quad \text{given by } z \mapsto x_v \cdot z,$$

is well-defined and is an isomorphism (of degree 1) of $k[x_1, \ldots, x_n]$-modules. Its inverse, $x_v \cdot z \mapsto z$, is given by replacing each occurrence of $x_v$ in $z$ with $-\theta'_1$ and setting all $x_j$ for $j \neq v$ not in the link of $v$ to zero.

Proof of Theorem 1.4: To prove the theorem it is enough to show that the socle of $\overline{k}(\Delta) = k(\Delta)/I$, where $I = \bigoplus_{j=0}^{d−1} \text{Soc} k(\Delta)_j$, vanishes in all degrees $j \neq d$. This is clear
for \( j = d - 1 \). For \( j \leq d - 2 \), consider any element \( y \in \mathbf{k}(\Delta)_j \) such that \( x_v \cdot y \in \text{Soc} \mathbf{k}(\Delta) \) for all \( v \in [n] \). We have to check that \( y \in \text{Soc} \mathbf{k}(\Delta) \). And indeed, the isomorphism of Theorem 3.1 implies that \( \overline{y} := \phi^{-1}(x_v \cdot y) \in \mathbf{k}(\text{lk} v)_j \) is in the socle of \( \mathbf{k}(\text{lk} v) \). Since \( \text{lk} v \) is a \((d - 2)\)-dimensional homology sphere, \( \mathbf{k}(\text{lk} v) \) is Gorenstein, and hence its socle vanishes in all degrees except \((d - 1)\)-st one. Therefore, \( \overline{y} = 0 \), and hence \( x_v \cdot y = \phi(\overline{y}) = 0 \) in \( \mathbf{k}(\Delta) \). Since this happens for all \( v \in [n] \), it follows that \( y \in \text{Soc} \mathbf{k}(\Delta) \). \( \square \)

We now turn to discussing the sphere and manifold \( g \)-conjectures and the connection between them. As was mentioned in the introduction, the strongest \( g \)-conjecture for homology spheres and its generalization (due to Kalai) for homology manifolds asserts that if \( \Delta \) is a \((d - 1)\)-dimensional connected orientable homology manifold, then for generically chosen \( \omega \) and \( \Theta = \{ \theta_1, \ldots, \theta_d \} \) in \( \mathbf{k}[\Delta]_1 \), the map

\[ \omega^{d-2i} : \mathbf{k}(\Delta)_i \rightarrow \mathbf{k}(\Delta)_{d-i} \]

is an isomorphism for all \( i \leq [d/2] \).

We refer to this conjecture as the strong (sphere or manifold) \( g \)-conjecture. If true, it would imply that \( \omega : \mathbf{k}(\Delta)_i \rightarrow \overline{\mathbf{k}(\Delta)}_{i+1} \) is injective for all \( i < [d/2] \) and is surjective for all \( i \geq [d/2] \). We refer to this weaker statement as the (sphere or manifold) \( g \)-conjecture. (Both, the stronger and the weaker conjectures yield exactly the same combinatorial restrictions on the face numbers of \( \Delta \).) Clearly, the manifold \( g \)-conjecture implies the sphere \( g \)-conjecture. The following result shows that they are almost equivalent: the strong sphere \( g \)-conjecture in the middle degree implies the manifold \( g \)-conjecture in all degrees.

**Theorem 3.2** Let \( \Delta \) be a \((d - 1)\)-dimensional connected orientable homology manifold. If for at least \((n - d)\) of the vertices \( v \) of \( \Delta \) and generically chosen \( \omega \) and \( \Theta' \) in \( \mathbf{k}[\text{lk} v]_1 \), the map \( \omega : \mathbf{k}(\text{lk} v)_{(d-1)/2} \rightarrow \mathbf{k}(\text{lk} v)_{(d-1)/2}+1 \) is surjective, then \( \Delta \) satisfies the manifold \( g \)-conjecture.

**Proof:** The condition on the links implies by [12, Theorem 4.26] that for generic choices of \( \omega \) and \( \Theta \) in \( \mathbf{k}[\Delta]_1 \), the map \( \omega : \mathbf{k}(\Delta)_{[d/2]} \rightarrow \mathbf{k}(\Delta)_{[d/2]+1} \) is surjective. Hence the map \( \omega : \overline{\mathbf{k}(\Delta)}_{[d/2]} \rightarrow \overline{\mathbf{k}(\Delta)}_{[d/2]+1} \) is surjective. Thus, \( \overline{\mathbf{k}(\Delta)}/\omega \) vanishes for \( i = [d/2] + 1 \), and hence also for all \( i > [d/2] + 1 \), and we infer that \( \omega : \overline{\mathbf{k}(\Delta)}_i \rightarrow \overline{\mathbf{k}(\Delta)}_{i+1} \) is surjective for \( i \geq [d/2] \). This in turn yields that the dual map \( \omega : \text{Hom}_\mathbf{k}(\overline{\mathbf{k}(\Delta)}_{i+1}, \mathbf{k}) \rightarrow \text{Hom}_\mathbf{k}(\overline{\mathbf{k}(\Delta)}_i, \mathbf{k}) \) is injective for all \( i \geq [d/2] \). Since \( \overline{\mathbf{k}(\Delta)} \) is Gorenstein, \( \text{Hom}_\mathbf{k}(\overline{\mathbf{k}(\Delta)}_i, \mathbf{k}) \) is naturally isomorphic to \( \overline{\mathbf{k}(\Delta)}_{d-i} \) (see Theorems I.12.5 and I.12.10 in [11]). Therefore, \( \omega : \overline{\mathbf{k}(\Delta)}_j \rightarrow \overline{\mathbf{k}(\Delta)}_{j+1} \) is injective for \( j < [d/2] \). \( \square \)

## 4 A special case of the \( g \)-theorem

In this section we prove Theorem 1.5. As in the proof of Theorem 1.4 we will rely on Theorem 3.1 and notation introduced there. Since the set of all \((\omega, \Theta)\) for which \( \omega^{d-2} : \mathbf{k}(\Delta)_1 \rightarrow \mathbf{k}(\Delta)_{d-1} \) is an isomorphism, is a Zariski open set (see [12, Section 4]), it
is enough to find one $\omega$ that does the job. Surprisingly, the $\omega$ we find is “very non-generic”: as we will see, $w = x_v$ where $v \in \tau$ does the job.

**Proof of Theorem 1.5:** We use induction on $d$ starting with $d = 3$. In this case, the face $\tau$ is simply a vertex, say $v$. Let $\sigma = \{v = v_1, v_2, v_3\}$ be a facet containing $v$. Let $\Theta$ be a generic l.s.o.p. for $k[\Delta]$, and as in Section 3 assume that $\theta_i = x_{v_i} + \sum_{j \notin \sigma} \theta_{i,j} x_j$. Since $\dim_k k[\Delta]_1 = \dim_k k[\Delta]_2$ (see Eq. (1)) and since $\dim \text{Soc} k[\Delta]_1 = 0$ (e.g. by Theorem 1.3), we will be done if we check that the map $\cdot x_v : k[\Delta]_1 \rightarrow k[\Delta]_2$ is injective. And indeed, if $x_v \cdot z \in \text{Soc} k[\Delta]_2$ for some $z \in k[\Delta]_1$, then by the isomorphism of Theorem 3.1, $\overline{z} = \phi^{-1}(x_v \cdot z)$ is in $\text{Soc} k(lk v)_1$. But the later space is the zero space, so $\overline{z} = 0$ in $k(lk v)_1$, that is, $\overline{z} \in \text{Span}(\theta_2', \theta_3')$. Since $lk v$ contains all the vertices of $\Delta$ except $v$, it follows that $\theta_2' = \theta_2$, $\theta_3' = \theta_3$, and $z - \overline{z}$ is a multiple of $\theta_1$, and hence that $z = 0$ in $k[\Delta]$.

The proof of the induction step goes along the same lines: let $\tau = \{v_1, \ldots, v_{d-2}\}$, let $\sigma = \tau \cup \{v_{d-1}, v_d\}$ be any facet containing $\tau$, and let $\Theta$ be a generic l.s.o.p. for $k[\Delta]$. We show that for $v \in \tau$, say $v = v_1$, the map $\cdot x^{d-2}_v : k[\Delta]_1 \rightarrow k[\Delta]_{d-1}$ is an injection. If $x_v^{d-2} \cdot z \in \text{Soc} k[\Delta]_{d-1}$ for some $z \in k[\Delta]_1$, then by Theorem 3.1,

$$x_v^{d-3} \cdot \overline{z} = (-\theta_1')^{d-3} \cdot \overline{z} \in \text{Soc} k(lk v)_{d-2}.$$

However, $lk v$ is a $(d-2)$-dimensional homology sphere satisfying the same assumptions as $\Delta$: the star of the face $\tau - \{v\}$ contains all the vertices of $lk v$. Hence by inductive hypothesis applied to $lk v$ with $\omega = -\theta_1'$ and $\Theta' = (\theta_2', \ldots, \theta_d')$, the map $\cdot (-\theta_1')^{d-3} : k(lk v)_1 \rightarrow k(lk v)_{d-2}$ is an injection. Thus $\overline{z} = 0$ in $k(lk v)_1$, and so $z = 0$ in $k[\Delta]_1$. □

**References**


