Chapter 2

STRAIGHTNESS ON SPHERES

> ... [I]t will readily be seen how much space lies between the two places themselves on the circumference of the large circle which is drawn through them around the earth. ... [W]e grant that it has been demonstrated by mathematics that the surface of the land and water is in its entirety a sphere, ... and that any plane which passes through the center makes at its surface, that is, at the surface of the earth and of the sky, great circles, and that the angles of the planes, which angles are at the center, cut the circumferences of the circles which they intercept proportionately, ...
>
> — Ptolemy, *Geographia* (ca. 150 A.D.) Book One, Chapter II

This chapter asks you to investigate the notion of straightness on a sphere, drawing on the understandings about straightness you developed in Problem **1.1**.

EARLY HISTORY OF SPHERICAL GEOMETRY

Observations of heavenly bodies were carried out in ancient Egypt and Babylon, mainly for astrological purposes and for making a calendar, which was important for organizing society. Claudius Ptolemy (c. 100–178), in his *Almagest*, cites Babylonian observations of eclipses and stars dating back to the 8h century B.C. The Babylonians originated the notion of dividing a circle into 360 degrees — speculations as to why 360 include that it was close to the number of days in a year, it was convenient to use in their hexadecimal system of counting, and 360 is the number of ways that seven points can be placed on a circle without regard to orientation (for the ancients there were seven "wandering bodies" — sun, moon, Mercury, Venus, Mars, Saturn, and Jupiter). But, more important, the Babylonians developed a coordinate system (essentially the same as what we now call "spherical coordinates") for the celestial sphere (the apparent sphere on which the stars, sun, moon, and planets appear to move) with its pole at the north star. Thus it is a misconception to think that the use of coordinates originated with Descartes in the 17th century.

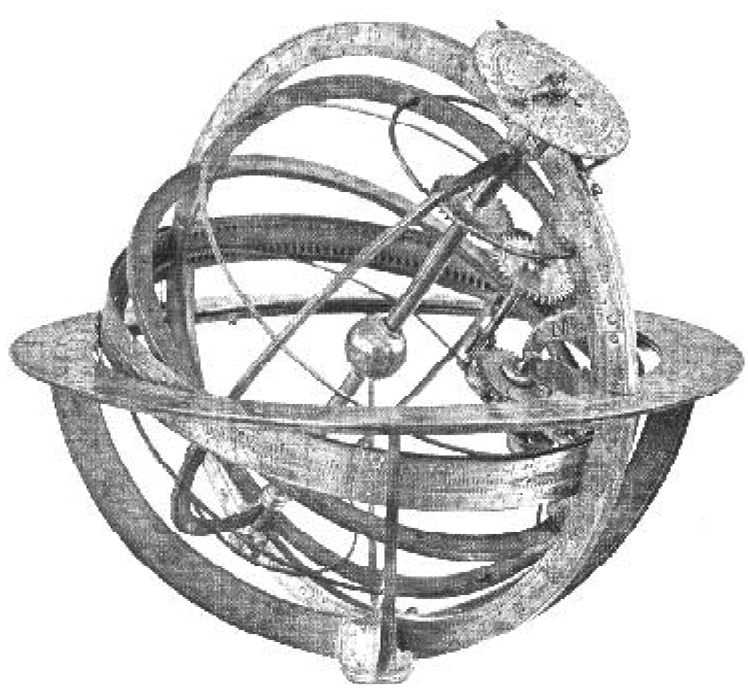


Figure 2.1 Armillary sphere (1687) showing (from inside out): earth, celestial sphere, ecliptic, and the horizon

The ancient Greeks became familiar with Babylonian astronomy around 4th century B.C. Eudoxus (408–355 B.C.) developed the "two-sphere model" for astronomy. In this model the stars are considered to be on the celestial sphere (which rotates one revolution a day westward about its pole, the north star) and the sun is on the sphere of the ecliptic, whose equator is the path of the sun and which is inclined to the equator of the celestial sphere at an angle that was about 24° in Eudoxus' time and is about 23½° now. The sphere of the ecliptic is considered to be attached to the celestial sphere and has an apparent rotation eastward of one revolution in a year. Both of these spheres appear to rotate about their poles. See Figure 2.1.

Autolycus, in *On the Rotating Spheres* (333–300 B.C.), introduced a third sphere whose pole is the point directly overhead a particular observer and whose equator is the visible horizon. Thus the angle between the horizon and the celestial equator is equal to the angle (measure at the center of the earth) between the observer and the north pole. Autolycus showed that, for a particular observer, some points (stars) of the celestial sphere are "always visible," some are "always invisible," and some "rise and set." See Figure 2.1.

The earliest known mathematical works that mention spherical geometry are Autolycus' book just mentioned and Euclid's *Phaenomena* [**AT:** Berggren] (300 B.C.). Both of these books use theorems from spherical geometry to solve astrological problems such as *What is the length of daylight on a particular date at a particular latitude?* Euclid used throughout definitions and propositions from spherical geometry. The definitions include *A great circle is the intersection of the sphere by a plane through its center* and *The intersection of the sphere by a plane not through the center forms a (small) circle that is parallel to a unique great circle*. The assumed propositions include, for example, *Suppose two circles are parallel to the same great circle C but on opposite sides; then the two circles are equal if and only if they cut off from some other great circle equal arcs on either side of C*. (We will see similar results in Chapter 10.) There are other more complicated results assumed, including one about the comparison of angles in a spherical triangle; see [**AT:** Berggren], page 25. Thus, it is implied by Autolycus' and Euclid's writings that there were previous works on spherical geometry available to their readers.

Hipparchus of Bithynia (190–120 B.C.) took the spherical coordinates of the Babylonians and applied them to the three spheres (celestial, ecliptic, and horizon). The solution to navigational and astrological problems (such as When will a particular star cross my horizon?) necessitated relating the coordinates on one sphere with the coordinates on the other spheres. This change of coordinates necessitates what we now call spherical trigonometry, and it appears that it was this astronomical problem with spherical coordinates that initiated the study of trigonometry. Plane trigonometry, apparently studied systematically first by Hipparchus, seems to have been originally developed in order to help with spherical trigonometry, which we will study in Chapter 20.

The first systematic account of spherical geometry was *Sphaerica* of Theodosius (around 200 B.C.) It consisted of three books of theorems and construction problems. Most of the propositions of *Sphaerica* were extrinsic theorems and constructions about a sphere as it sits with its center in Euclidean 3-space; but there were also propositions formulated in terms of the intrinsic geometry on the surface of a sphere without reference to either its center or 3-space. We will discuss the distinction between intrinsic and extrinsic later in this chapter.

A more advanced treatise on spherical trigonometry was *On the Sphere* by Menelaus (about 100 A.D.) There exist only edited Arabic versions of this work. In the introduction Menelaus defined a spherical triangle as part of a spherical surface bounded by three arcs of great circles, each less than a semicircle; and he defined the angles of these triangles. Menelaus' treatise expounds geometry on the surface of a sphere in a way analogous to Euclid's exposition of plane geometry in his *Elements*.

Ptolemy (100–178 A.D.) worked in Alexandria and wrote a book on geography, *Geographia* (quoted at the beginning of this chapter), and *Mathematiki Syntaxis* (*Mathematical Collections*), which was the result of centuries of knowledge from Babylonian astronomers and Greek geometers. It became the standard Western work on mathematical astronomy for the next 1400 years. The *Mathematiki Syntaxis* in generally known as the *Almagest*, which is a Latin distortion of the book's name in Arabic that was derived from one of its Greek names. The *Almagest* is important because it is the earliest existing work containing a study of spherical trigonometry, including specific functions, inverse functions, and the computational study of continuous phenomena.

More aspects of the history of spherical geometry will appear later in this book in the appropriate places. For more readings (and references to the primary literature) on this history, see [**HI**: Katz], Chapter 4, and [**HI**: Rosenfeld], Chapter 1.

PROBLEM 2.1 WHAT IS STRAIGHT ON A SPHERE?

Drawing on the understandings about straightness you developed in Problem **1.1**, this problem asks you to investigate the notion of straightness on a sphere. It is important for you to realize that, if you are not building a notion of straightness for yourself (for example, if you are taking ideas from books without thinking deeply about them), then you will have difficulty building a concept of straightness on surfaces other than a plane. Only by developing a personal meaning of straightness for yourself does it become part of your active intuition. We say *active* intuition to emphasize that intuition is in a process of constant change and enrichment, that it is not static.

a. *Imagine yourself to be a bug crawling around on a sphere.* (*This bug can neither fly nor burrow into the sphere.*) *The bug's universe is just the surface*; *it never leaves it. What is* "*straight*" *for this bug*? *What will the bug see or experience as straight*? *How can you convince yourself of this*? *Use the properties of straightness* (*such as symmetries*) *that you talked about in Problem* **1.1**.

b. *Show* (*that is, convince yourself, and give an argument to convince others*) *that the great circles on a sphere are straight with respect to the sphere, and that no other circles on the sphere are straight with respect to the sphere.*

Suggestions

Great circles are those circles that are the intersection of the sphere with a plane through the center of the sphere. Examples include longitude lines and the equator on the earth. Any pair of opposite points can be considered as the poles, and thus the equator and longitudes with respect to any pair of opposite points will be great circles. See Figure 2.2.

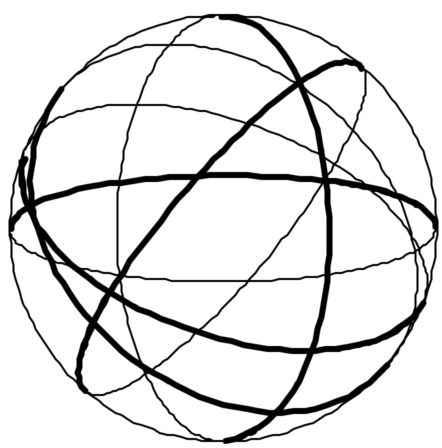


Figure 2.2 Great circles

The first step to understanding this problem is to convince yourself that great circles are straight lines on a sphere. Think what it is about the great circles that would make the bug experience them as straight. To better visualize what is happening on a sphere (or any other surface, for that matter), **you must use models**. This is a point we cannot stress enough. The use of models will become increasingly important in later problems, especially those involving more than one line. You must make lines on a sphere to fully understand what is straight and why. An orange or an old, worn tennis ball work well as spheres, and rubber bands make good lines. Also, you can use ribbon or strips of paper. Try placing these items on the sphere along different curves to see what happens.

Also look at the symmetries from Problem **1.1** to see if they hold for straight lines on the sphere. The important thing here is to **think in terms of the surface of the sphere, not the solid 3-dimensional ball**. Always try to imagine how things would look from the bug's point of view. A good example of how this type of thinking works is to look at an insect called a water strider. The water strider walks on the surface of a pond and has a very 2-dimensional perception of the world around it — to the water strider, there is no up or down; its whole world consists of the 2-dimensional plane of the water. The water strider is very sensitive to motion and vibration on the water's surface, but it can be approached from above or below without its knowledge. Hungry birds and fish take advantage of this fact. This is the type of thinking needed to visualize adequately properties of straight lines on the sphere. For more discussion of water striders and other animals with their own varieties of intrinsic observations, see the delightful book *The View from the Oak*, by Judith and Herbert Kohl [**NA**: Kohl and Kohl].

> **Definition.** Paths that are intrinsically straight on a sphere (or other surfaces) are called ***geodesics***.

This leads us to consider the concept of intrinsic or geodesic curvature versus extrinsic curvature. As an outside observer looking at the sphere in 3-space, all paths on the sphere, even the great circles, are curved — that is, they exhibit ***extrinsic curvature***. But relative to the surface of the sphere (*intrinsically*), the lines may be straight and thus have intrinsic curvature zero. See the last section of this chapter, Intrinsic Curvature. Be sure to understand this difference and to see why all symmetries (such as reflections) must be carried out intrinsically, or from the bug's point of view.

It is natural for you to have some difficulty experiencing straightness on surfaces other than the 2-dimensional plane; it is likely that you will start to look at spheres and the curves on spheres as 3-dimensional objects. Imagining that you are a 2-dimensional bug walking on a sphere helps you to shed your limiting extrinsic 3-dimensional vision of the curves on a sphere and to experience straightness intrinsically. Ask yourself the following:

- What does the bug have to do, when walking on a non-planar surface, in order to walk in a straight line?
- How can the bug check if it is going straight?

Experimentation with models plays an important role here. Working with models that *you create* helps you to experience great circles as, in fact, the only straight lines on the surface of a sphere. Convincing yourself of this notion will involve recognizing that straightness on the plane and straightness on a sphere have common elements. When you are comfortable with "great-circle-straightness," you will be ready to transfer the symmetries of straight lines on the plane to great circles on a sphere and, later, to geodesics on other surfaces. Here are some activities that you can try, or visualize, to help you experience great circles and their intrinsic straightness on a sphere. However, it is better for you to come up with your own experiences.

- Stretch something elastic on a sphere. It will stay in place on a great circle, but it will not stay on a small circle if the sphere is slippery. Here, the elastic follows a path that is approximately the shortest because a stretched elastic always moves so that it will be shorter. This a very useful practical criterion of straightness.
- Roll a ball on a straight chalk line (or straight on a freshly painted floor!). The chalk (or paint) will mark the line of contact on the sphere, and it will form a great circle.
- Take a narrow stiff ribbon or strip of paper that does not stretch, and lay it "flat" on a sphere. It will only lie (without folds and creases) along a great circle. Do you see how this property is related to local symmetry? This is sometimes called the *Ribbon Test*. (For further discussion of the Ribbon Test, see Problems **3.4** and **7.6** of [**DG**: Henderson].)

- The feeling of turning and "nonturning" comes up. Why is it that on a great circle there is no turning and on a latitude line there is turning? Physically, in order to avoid turning, the bug has to move its left feet the same distance as its right feet. On a non-great circle (for example, a latitude line that is not the equator), the bug has to walk faster with the legs that are on the side closer to the equator. This same idea can be experienced by taking a small toy car with its wheels fixed to parallel axes so that, on a plane, it rolls along a straight line. On a sphere, the car will roll around a great circle; but it will not roll around other curves.
- Also notice that, on a sphere, straight lines are intrinsic circles (points on the surface a fixed distance along the surface away from a given point on the surface) — special circles whose circumferences are straight! Note that the equator is a circle with two intrinsic centers: the north pole and the south pole. In fact, any circle (such as a latitude circle) on a sphere has two intrinsic centers.

These activities will provide you with an opportunity to investigate the relationships between a sphere and the geodesics of that sphere. Along the way, your experiences should help you to discover how great circles on a sphere have most of the same symmetries as straight lines on a plane.

You should pause and not read further until you have expressed your thinking and ideas about this problem.

Symmetries of Great Circles

Reflection-through-itself symmetry: We can see this globally by placing a hemisphere on a flat mirror. The hemisphere together with the image in the mirror exactly recreates a whole sphere. Figure 2.3 shows a reflection through the great circle g.

Reflection-perpendicular-to-itself symmetry: A reflection through any great circle will take any great circle (for example, g' in Figure 2.3) perpendicular to the original great circle onto itself.

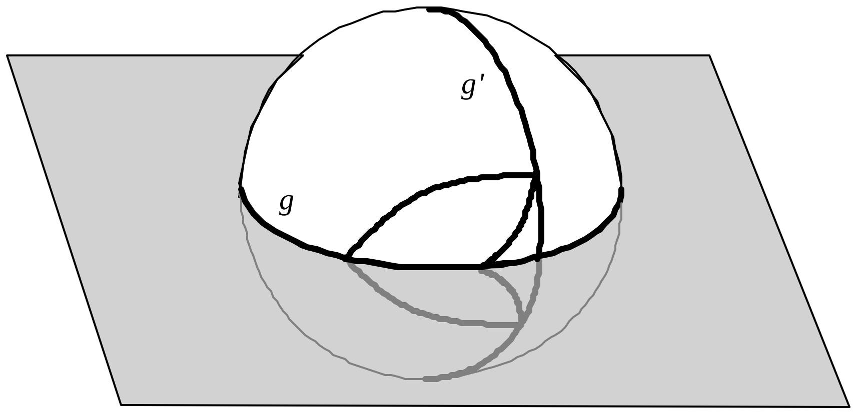


Figure 2.3 Reflection-through-itself symmetry

Half-turn symmetry: A rotation through half of a full revolution about any point *P* on a great circle interchanges the part of the great circle on one side of *P* with the part on the other side of *P*. See Figure 2.4.

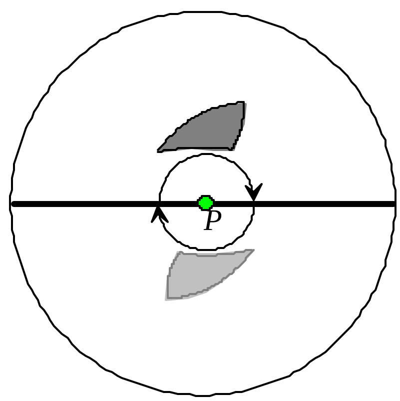


Figure 2.4 Half-turn symmetry

Rigid-motion-along-itself symmetry: For great circles on a sphere, we call this a translation along the great circle or a rotation around the poles of that great circle. This property of being able to move rigidly along itself is not unique to great circles because any circle on the sphere will also have the same symmetry. See Figure 2.5.

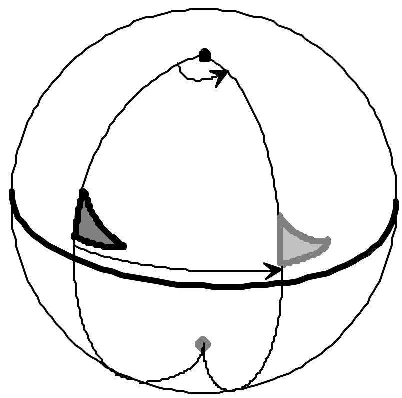
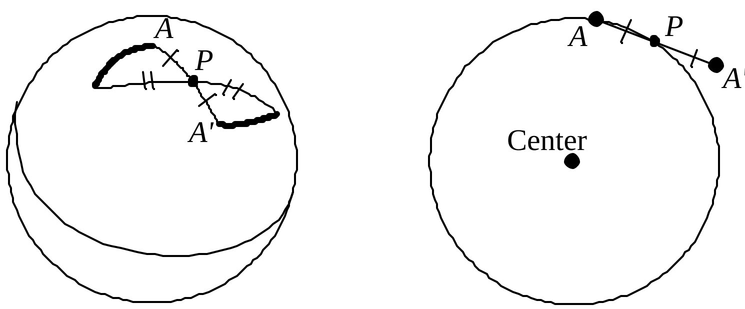


Figure 2.5 Rigid-motion-along-itself symmetry

Central symmetry, or point symmetry: Viewed intrinsically (from the 2-dimensional bug's point-of-view), central symmetry through a point *P* on the sphere sends any point *A* to the point at the same great circle distance from *P* but on the opposite side. See Figure 2.6.



Intrinsically Extrinsically

Figure 2.6 Central symmetry through *P*

Extrinsically (viewing the sphere in 3-space) central symmetry through *P* would send *A* to a point off the surface of the sphere as shown in Figure 2.6. The only extrinsic central symmetry of the sphere (and the only one for great circles on the sphere) is through the center of the sphere (which is not *on* the sphere). The transformation that is intrinsically central symmetry is extrinsically half-turn symmetry (about the diameter through *P*). Intrinsically, as on a plane, central symmetry does not differ from half-turn symmetry with respect to the end result. This distinction between intrinsic and extrinsic is important to experience at this point.

3-dimensional-rotation symmetry: This symmetry does not hold for great circles in 3-space; however, it does hold for great circles in a 3-sphere. See Problem **22.5**.

You will probably notice that other objects on the sphere, besides great circles, have some of the symmetries mentioned here. It is important for you to construct such examples. This will help you to realize that straightness and the symmetries discussed here are intimately related.

*Every Geodesic Is a Great Circle

Notice that you were not asked to prove that *every geodesic* (intrinsic straight line) *on the sphere is a great circle*. This is true but more difficult to prove. Many texts simply *define* the great circles to be the "straight lines" (geodesics) on the sphere. We have not taken that approach. We have shown that the great circles are intrinscially straight (geodesics), and it is clear that two points on the sphere are always joined by a great circle arc, which shows that there are sufficient great- circle geodesics to do the geometry we wish.

To show that great circles are the only geodesics involves some notions from differential geometry. In Problem **3.2b** of [**DG**: Henderson] this is proved using special properties of plane curves. More generally, a geodesic satisfies a differential equation with the initial condition being a point on the geodesic and the direction of the geodesic at that point (see Problem **8.4b** of [**DG**: Henderson]). Thus it follows from the analysis theorem on the *existence and uniqueness of solutions to differential equations* that

> **Theorem 2.1.** *At every point and in every direction on a smooth surface there is a unique geodesic going from that point in that direction.*

From this it follows that all geodesics on a sphere are great circles. *Do you see why?*

*Intrinsic Curvature

You have tried wrapping the sphere with a ribbon and noticed that the ribbon will only lie flat along a great circle. (If you haven't experienced this yet, then do it now before you go on.) Arcs of great circles are the only paths on a sphere's surface that are tangent to a straight line on a piece of paper wrapped around the sphere.

If you wrap a piece of paper tangent to the sphere around a latitude circle (see Figure 2.7), then, extrinsically, the paper will form a portion of a cone and the curve on the paper will be an arc of a circle when the paper is flattened. The *intrinsic curvature* of a path on the surface of a sphere can be defined as the curvature (1/radius) that one gets when one "unwraps" the path onto a plane. For more details, see Chapter 3 of [**DG**: Henderson].

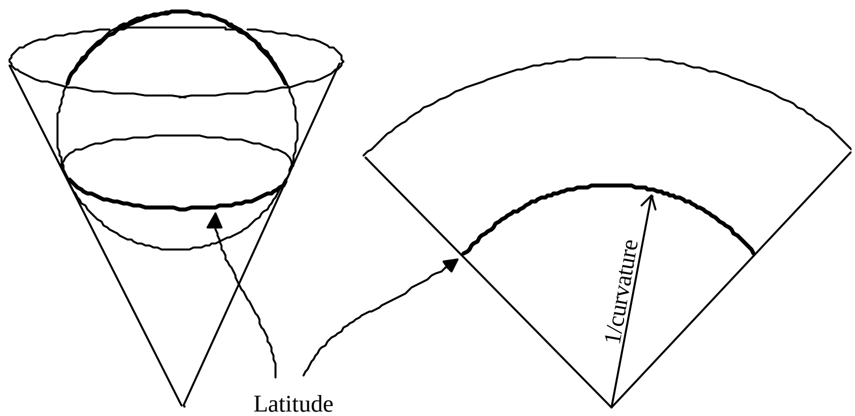


Figure 2.7 Finding the intrinsic curvature

Differential geometers often talk about intrinsically straight paths (geodesics) in terms of the velocity vector of the motion as one travels at a constant speed along that path. (The velocity vector is tangent to the curve along which the bug walks.) For example, as you walk along a great circle, the velocity vector to the circle changes direction, extrinsically, in 3-space where the change in direction is toward the center of the sphere. "Toward the center" is not a direction that makes sense to a 2-dimensional bug whose whole universe is the surface of the sphere. Thus, the bug does not experience the velocity vectors as changing direction at points along the great circle; however, along non-great circles the velocity vector will be experienced as changing in the direction of the closest center of the circle. In differential geometry, the rate of change, from the bug's point of view, is called the *covariant* (or *intrinsic*) *derivative*. As the bug traverses a geodesic, the covariant derivative of the velocity vector is zero. This can also be expressed in terms of *parallel transport*, which is discussed in Chapters 7, 8, and 10 of this text. See [**DG**: Henderson] for discussions of these ideas in differential geometry.