Occupation time fluctuations of branching processes with immigration

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Outline

Branching particle systems intuitively

What is a branching particle system?

A set of particles of some types, distributed somehow, somewhere, moving somehow and branching from time to time in a particular manner...

Settings of the system

- Particles - many types of particles (eg. dust particles)
- Space - most often $\mathbb{R}^d$ or $\mathbb{Z}^d$, generally a Polish space
- Time - discrete or continuous
- Evolution - particles evolve independently according to a time homogeneous Markov process (eg. Lévy process)
- Branching - after some time (exponential/geometric) a particle splits producing random number of offspring
- Immigration - new particles appear in the system in random fashion
- System as a whole is usually a measure-valued stochastic process

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Related concepts ...

- (Functional) limit theorems, superprocesses
- Occupation time and its fluctuations
- Interplay with the PDE's via a renewal argument and the Feynman-Kac formula

$$\mathcal{C}([0,1],X) \quad \mathcal{D}([0,1],X)$$

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$$X_t = \int_0^t N_s ds \left[- \int_0^t \mathbb{E} N_s ds \right]$$

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$$\frac{\partial h}{\partial t} = (\mathcal{A} - 1)h + F(h)$$

Settings

Description of the system - critical branching

- particles evolve according to a symmetric α-stable Lévy motion
- branching after exponential time
- branching law critical, binary
- starting distribution homogeneous Poisson random field
- immigration according to homogeneous Poisson random field in space-time

$$F(s) = \frac{1}{2}s^2 + \frac{1}{2}.$$

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- particles evolve according to a symmetric α-stable Lévy motion
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Investigated object: functional limit (under rescaling of time and space) for the occupation time fluctuations process

A functional limit theorem I - critical case

Theorem

Assumptions

- *space* $\mathbb{R}^d$, $F_T = T^{(4+d/\alpha)/2}$ *and* $0 < d < 2\alpha$

then

rescaled occupation time fluctuations process converges in $\mathcal{C}\left([0,\tau], \mathcal{S}'\left(\mathbb{R}^d\right)\right)$

$$X_T \to_c \lambda\eta, \quad \text{as } T \to +\infty,$$

where η *is centered Gaussian process with covariance*

$$c_h^\eta = s^{h+1} + t^{h+1} - \frac{1}{4}(s+t)^{h+1} - \frac{1}{4}(t-s)^h[3t + (2h-1)s],$$

$$h = (3 - \frac{d}{\alpha})/2.$$

A functional limit theorem II - critical case

Theorem

Assumptions

- *space* $\mathbb{R}^d$, $F_T = T$ *and* $2\alpha < d$

then

rescaled occupation time fluctuations process converges in $\mathcal{C}\left([0,\tau], \mathcal{S}'\left(\mathbb{R}^d\right)\right)$ *to* $\mathcal{S}'(\mathbb{R}^d)$*-valued process with covariance functional*

$$Cov\left(\langle X_s, \varphi_1\rangle, \langle X_t, \varphi_2\rangle\right) = \frac{(s\wedge t)^2}{2(2\pi)^d}\int_{\mathbb{R}^d}\left(\frac{2}{|z|^\alpha} + \frac{V}{|z|^{2\alpha}}\right)\widehat{\varphi}_1(z)\overline{\widehat{\varphi}_2(z)}dz.$$

Settings

Description of the system - subcritical branching

- particles evolve according to a Markov process (under mild assumptions)
- branching after exponential time
- branching law subcritical, binary
- starting distribution homogeneous Poisson random field
- immigration according to homogeneous Poisson random field in space-time

$$F(s) = qs^2 + (1-q), \quad q < 1/2.$$

Settings

Description of the system - subcritical branching

- particles evolve according to a Markov process (under mild assumptions)
- branching after exponential time
- branching law subcritical, binary
- starting distribution homogeneous Poisson random field
- immigration according to homogeneous Poisson random field in space-time

Investigated object: functional limit (under rescaling of time and space) for the occupation time fluctuations process

A limit theorem III - subcritical case

Theorem

Assumptions

- *space* $\mathbb{R}^d$, $F_T = T^{1/2}$

then

rescaled occupation time fluctuations process converges to $\mathcal{S}'(\mathbb{R}^d)$*-valued process with covariance*

$$Cov\left(\langle X_s, \varphi_1\rangle, \langle X_t, \varphi_2\rangle\right) = (s \wedge t) \int_{\mathbb{R}^d} T(\varphi_1, \varphi_2) dx.$$

where

$$T(\varphi) := U^Q\left(\varphi(x) U^Q \varphi(x)\right) + \int_0^{+\infty} U^Q(\mathcal{T}_t^Q \varphi(x) \mathcal{T}_t^Q U^Q \varphi(x)) dt,$$

$$T(\varphi_1, \varphi_2) = \frac{1}{2}\left(T(\varphi_1 + \varphi_2) - T(\varphi_1) - T(\varphi_2)\right).$$

For Further Reading

Luis G. Gorostiza, Reyla Navarro, and Eliane Rodrigues.
Some long-range dependence processes arising from fluctuations of particle systems.
Acta Appl. Math., 86:285–308(24), 2005.

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Large and moderate deviations for occupation times of immigration superprocesses.
Inf. Dimen. Anal., Quant. Probab. Rel. Top., 8(5):593–603, 2005.

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Occupation time fluctuations of poisson and equilibrium branching systems in critical and large dimensions.
Probab. and Math. Stat., to appear.

The end

Thank you!