Chaos in a Spatial Epidemic Model

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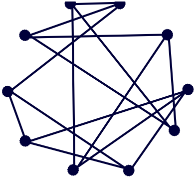
The process

Motivation: gypsy moth infected by the nuclear polyhedrosis virus.

Let G_N be a random 3-regular graph with N nodes. The state of each site is 0 = vacant or 1 = occupied.
The dynamics occur in discrete time with two alternate processes:



- Growth: No plants survive, and each one gives birth to a Poisson$[\beta]$ number of individuals which are each sent to a location chosen uniformly from the entire graph G_N.
- Epidemic: Each site is infected with a small probability α_N. If the site is occupied, the plant dies and the epidemic spreads to all neighboring (occupied) sites. We take $\alpha_N \to 0$.

We denote the process by $\eta_k^N \in \{0, 1\}^{G_N}$.

Mean field equations for the occupation density

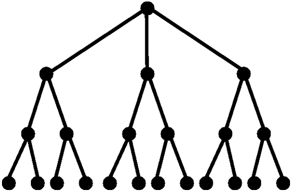
Growth: $p \longrightarrow f_N(p) = 1 - \left(1 - \frac{\beta}{N}\right)^{pN} \approx f(p) = 1 - e^{-\beta p}$.

Epidemic: In the limit $N \to \infty$, only the giant component of the set of occupied sites is destroyed by the epidemic. The graph looks locally like a regular tree of degree 3, so

$$\mathbb{P}(0 \text{ survives the epidemic})$$
$$= \mathbb{P}(0 \notin \text{giant component in a site percolation on the 3-tree})$$

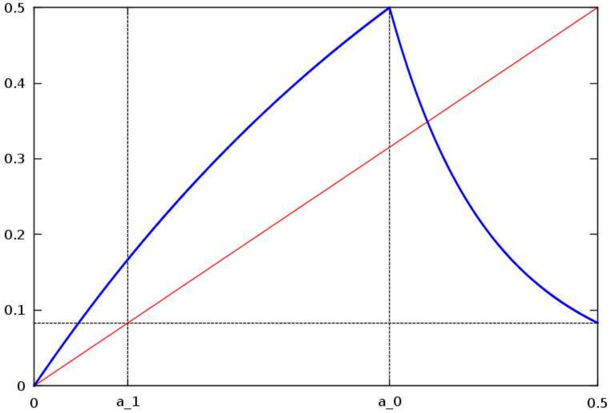
A branching process calculation allows to compute this probability, and we get for the density:

$$q \longrightarrow g(q) = \begin{cases} q & \text{if } q \in [0, 1/2], \\ \frac{(1-q)^3}{q^2} & \text{if } q \in (1/2, 1]. \end{cases}$$

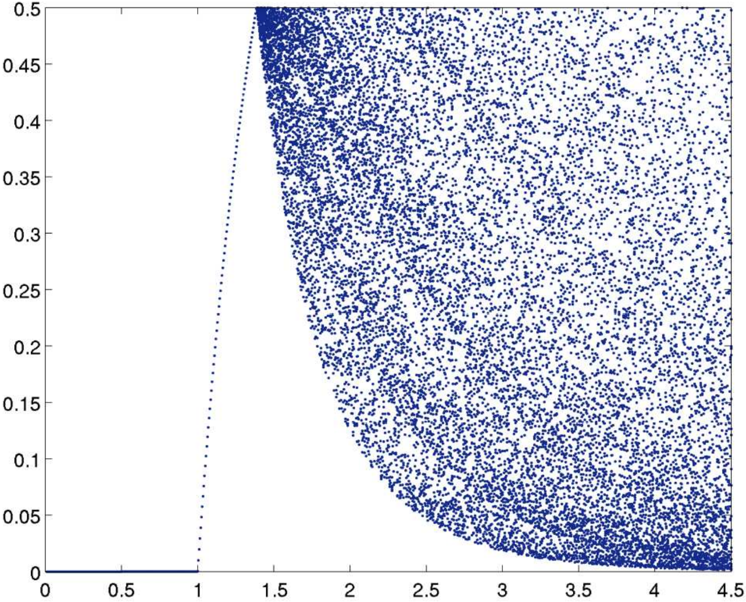


Mean field dynamics: letting $a_0 = f^{-1}(1/2) = (\ln 2)/\beta$:

$$h(p) = g(f(p)) = \begin{cases} 1 - e^{-\beta p} & 0 \le p \le a_0 \\ \frac{e^{-3\beta p}}{(1-e^{-\beta p})^2} & a_0 < p \le 1 \end{cases}.$$



Chaotic behavior of the iterates of h



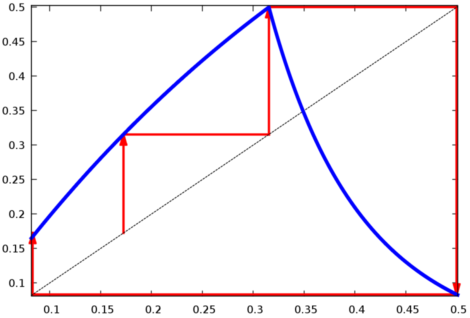
Theorem 1

(a) *The dynamical system defined by the function* $h\colon [a_1, 1/2] \longrightarrow [a_1, 1/2]$ *is chaotic for every* $\beta > 2\log 2$.

(b) *If* $\beta \in (2\log 2, 2.48]$ *then the system has an invariant measure which is absolutely continuous with respect to the Lebesgue measure.*

For (a) it is enough to prove by Li and Yorke (1975) that

$$\exists c \in [a_1, 1/2] \text{ such that } h^3(c) \leq c < h(c) < h^2(c).$$



Convergence of the process to the chaotic system

We let $\rho_k^N = \frac{1}{N}\sum_{i=1}^N \eta_k^N(i)$ and $\alpha_N = 1/\log_2 N$.

Theorem 2

The process $(\rho_k^N)_{k\geq 0}$ converges in distribution to the orbits of h.

Idea of the proof:

- Look at the epidemic ignoring infections from a distance greater than $\log_2 N/5$.
- At this scale, G_N looks like a 3-tree with high probability.
- Couple the restricted process with a process constructed from a site percolation on the infinite 3-tree.

The process on the 2-dimensional torus

We now let $G_N = (\mathbb{Z} \mod N)^2$ $(\{1, \ldots, N\}^2$ with periodic boundary).

Repeating the proof of Theorem 2 we get convergence of the sequence of densities to an analogous system $h = g \circ f$.

Here f remains the same, while

$$g(p) = p - \mathbb{P}_p\big(0 \text{ percolates in a site percolation in } \mathbb{Z}^2\big) .$$

Is the system chaotic? It seems so, for $\beta > \beta_c = \frac{1}{p_c} \log\left(\frac{1}{1-p_c}\right) \approx 1.516$.

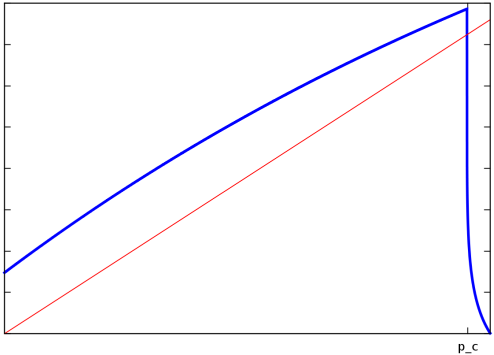
Theorem 3

There is an $\varepsilon > 0$ such that for every $\beta \in (\beta_c, \beta_c + \varepsilon)$ the dynamical system $(h^k(p))_{k \geq 0}$ has an invariant measure which is absolutely continuous with respect to the Lebesgue measure.

To prove this we use the fact (Kesten and Zhang (1987)) that for some $\gamma \in (0,1)$, $C > 0$,

$$\mathbb{P}_p\big(0 \text{ percolates in a site percolation in } \mathbb{Z}^2\big) \geq C(p - p_c)^\gamma$$

for $p \sim p_c$, $p > p_c$.



$$(h^k)'(p) = h'(h^{k-1}(p))h'(h^{k-2}(p)) \cdots h'(p).$$

Local growth on the 2-dimensional torus

We finish by introducing local interactions in the growth step: each plant at a site i sends a Poisson$[\beta]$ number of children to locations chosen uniformly from $B(i, r_N)$.

We take $r_N \to \infty$, $\frac{r_N}{N} \to 0$, and $\alpha_N r_N \to \infty$.

Theorem 4

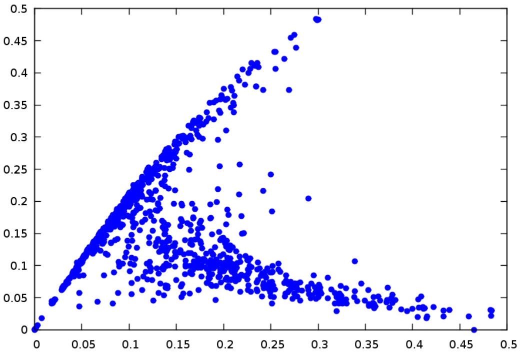
$$\mathbb{P}\Big(\eta_k^N(0) = 1\Big) = \mathbb{E}\Big(\rho_k^N\Big) \xrightarrow[N\to\infty]{} h^k(p)$$

for every $k \geq 0$. *In particular, given any* $K \in \mathbb{N}$, *the sequence of expected densities* $\left(\mathbb{P}(0 \in \eta_k^N)\right)_{k=1,\dots,K}$ *converges to* $\left(h^k(p)\right)_{k=1,\dots,K}$.

Idea of the proof:

- With high probability, after running the growth and epidemic steps, the fraction of occupied sites on $B(i, r_N)$ for each i is close to uniform.
- Look at the process ignoring infections coming from distance greater than $l_N \ll r_N$.
- Compare with the process with $r_N = \infty$.
- Coupling...

The theorem does not seem to be true if we keep r_N as a fixed constant.



Graph of (ρ_k^N, ρ_{k+1}^N). Here $N = 750$ and $r_N = 50$.