Ecological equilibrium for Restrained Branching Random Walks

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Motivation

We want to construct a model for populations where individuals have a random number of sons/daughters.

Here, for instance, we have one father,

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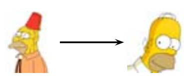

Here, for instance, we have one father, one son

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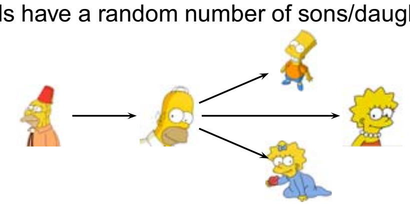

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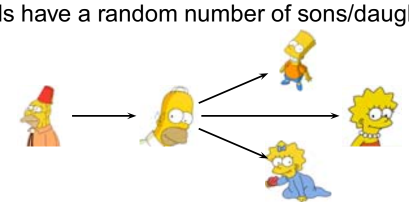

Here, for instance, we have one father, one son and three grandchildren.

Restriction in the model:

each individual has exactly one parent (i.e. no sexual mating, think for instance of bacteria).

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2. each individual breeds independently of all others,
3. with the same offspring distribution (let μ be its expected value).

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The Galton-Watson process: example

generation 0
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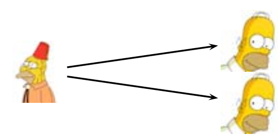

generation 0 — 1 individual
generation 1 — 2 individuals

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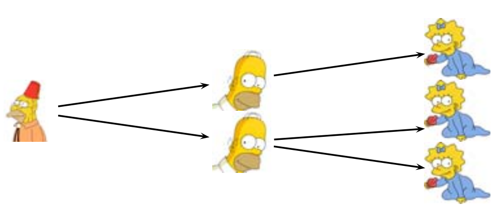

generation 0 — 1 individual
generation 1 — 2 individuals
generation 2 — 3 individuals

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The Galton-Watson process: the phases

Only two possible behaviours/phases

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- $\mu > 1 \Longrightarrow$ the colony survives with positive probability. Conditioned to survival, the number of individuals tends to ∞.

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If we require a maximal size for the colony

Note that the requirement brings interaction in the breeding mechanism.
By finite state Markov chain arguments, we have almost sure extinction for all μ.

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Continuous time Branching Process

In this model:

1. time is continuous;

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In this model:

1. time is continuous;
2. each individual has an *Exp*(1) lifetime;

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Continuous time Branching Process

In this model:

1. time is continuous;
2. each individual has an *Exp*(1) lifetime;
3. each individual has a *breeding* Poisson clock with parameter λ; when this clock beats, the individual generates a son.

Note

Generations overlap in time.

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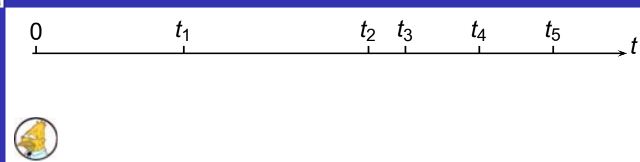
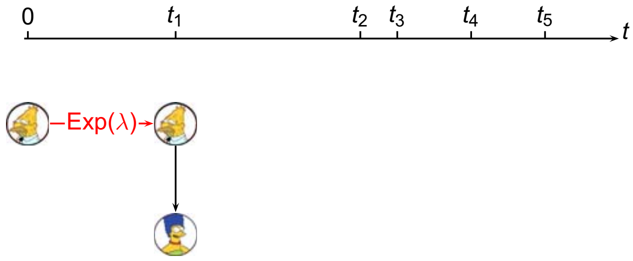
CT BP: an example



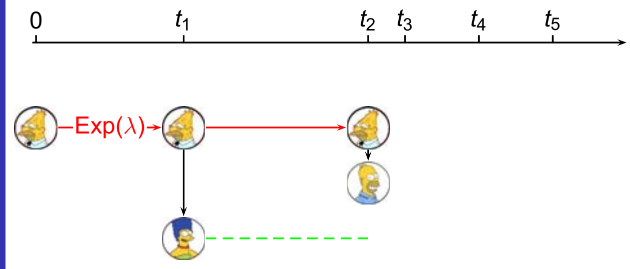
At time 0 there is one individual.

CT BP: an example



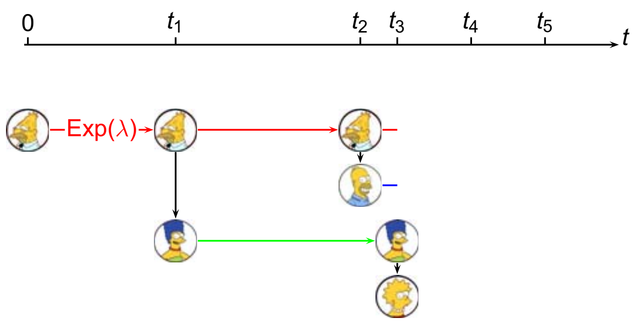
At time t_1 ($t_1 \sim Exp(\lambda)$) a son is born.

CT BP: an example



After another exponential waiting time, at time t_2 another son is born.

CT BP: an example



At time t_3 the first son has a son ($t_3 - t_1 \sim Exp(\lambda)$).

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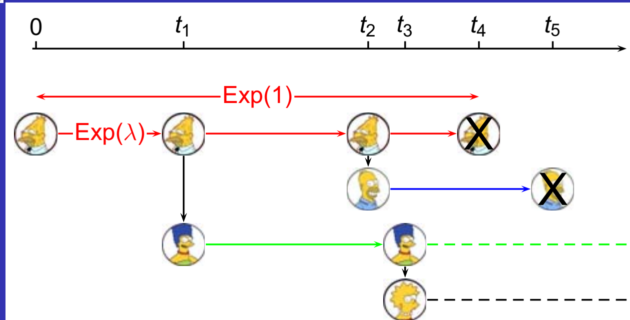

At time t_4 ($t_4 \sim Exp(1)$) the ancestor dies.
At time t_5 the second son dies, and so at time $t > t_5$ there are 2 individuals alive...and so on.

CT BP: the phases

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We have almost sure extinction for all λ.

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In order to create a richer model, one adds space.

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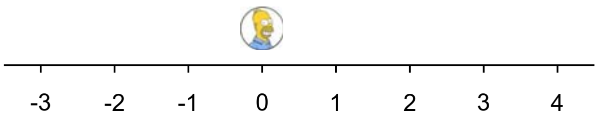

Here $X = \mathbb{Z}$ and P is the simple random walk. At time 0 there is one individual living at site 0.

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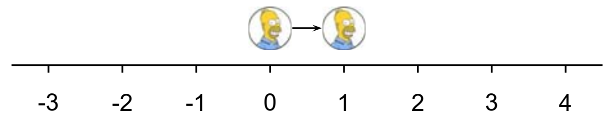

Here $X = \mathbb{Z}$ and P is the simple random walk. At time 0 there is one individual living at site 0.
After an $Exp(\lambda)$ time, it generates a son which flips a coin to decide where to go and finally goes to site 1.

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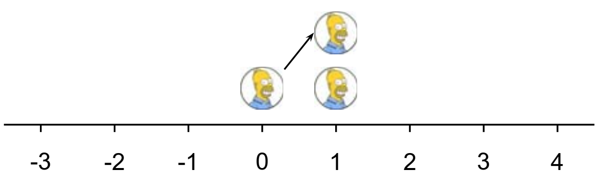

After another $Exp(\lambda)$ time, another son is generated and placed in site 1.

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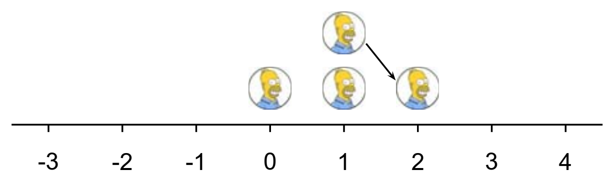

The second son has a son in site 2.

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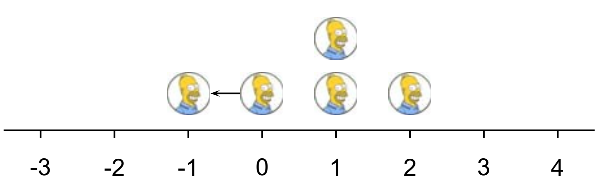

The first individual has a third son

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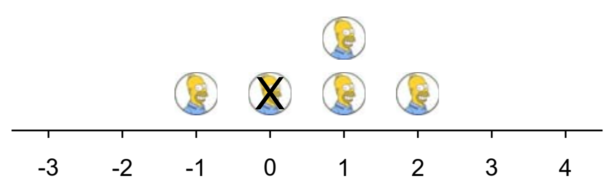

The first individual has a third son, and (after a time $Exp(1)$ from the start) dies. The offsprings breed and die with the same laws and the process continues.

Branching Random Walk: the phases

Only two possible behaviours/phases

For the BRW *starting with one individual* at a fixed site x_0, there exists $\lambda_c > 0$ such that:

- $\lambda < \lambda_c \Longrightarrow$ almost sure extinction;

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- $\lambda < \lambda_c \Longrightarrow$ almost sure extinction;
- $\lambda > \lambda_c \Longrightarrow$ the colony survives with positive probability. Conditioned to survival, the expected number of individuals per site tends to ∞.
- If we require a maximal size M for the population in each site, then $\lambda_c = \lambda_c(M)$ and we still have the two phases described above (if $|X| = \infty$).

Examples

- If $X = \mathbb{Z}^d$ and P = simple random walk $\Longrightarrow \lambda_c = 1$.

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Remark: the weak phase

Depending on $(X, E(X))$ and on the transition kernel P, the survival phase may be further subdivided into a weak (survival but no return to x_0) and a strong one (survival returning to x_0).

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Here we will not deal with this subject – we restrict ourselves to $X = \mathbb{Z}^d$ with P = simple random walk (where there is no weak phase).

Other issues

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Our goal

We look for a model where
- there are no *a priori* bounds on the local size of the population;

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Idea

Introduce a self-regulating mechanism in the births. If a site is crowded, newcomers are penalized.

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Restrained Branching Random Walk

We will stick to $X = \mathbb{Z}^d$ and P = the simple random walk. Choose $c : \mathbb{N} \to [0, +\infty)$ a nonincreasing function such that $c(0) = \lambda$.

RBRW

1. each individual has an $Exp(1)$ lifetime;

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1. each individual has an $Exp(1)$ lifetime;
2. each individual has a Poisson clock with parameter λ; when this clock beats, the individual generates a son;
3. the son of an individual living at x, once generated, chooses a location where to spend its life according to a probability distribution $p(x, \cdot)$;
4. if the location is y and there are k individuals at y, the birth is eliminated with probability $1 - \frac{c(k)}{\lambda}$.

The role of c

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The function c takes into account the fierceness of the competition for resources.

Indeed if there are k individual in a given site, newcomers are eliminated with probability $1 - \frac{c(k)}{\lambda}$.

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- If $c(k) = \lambda$ for all $k \le k_0$, then at a given site there are no eliminations until the population reaches $k_0 + 1$ individuals.
- If $c(k) = 0$ for all $k \ge k_0$, then there may be at most k_0 individuals at each site.

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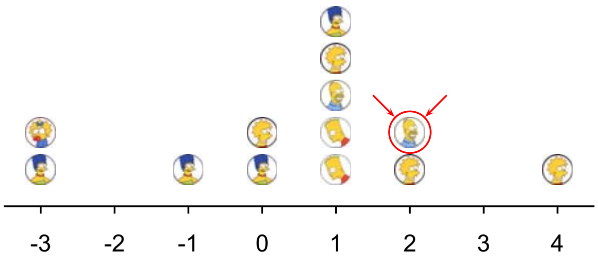
- If $c(k) = \lambda$ for all $k \le k_0$, then at a given site there are no eliminations until the population reaches $k_0 + 1$ individuals.
- If $c(k) = 0$ for all $k \ge k_0$, then there may be at most k_0 individuals at each site.
- If $c(k) > 0$ for all k, but $c(k) \stackrel{k\to\infty}{\to} 0$, the faster it tends to zero, the more difficult it is to be born at crowded sites.

Restrained BRW: an example



Suppose that at this epoch the individual encircled in red generates a new individual.

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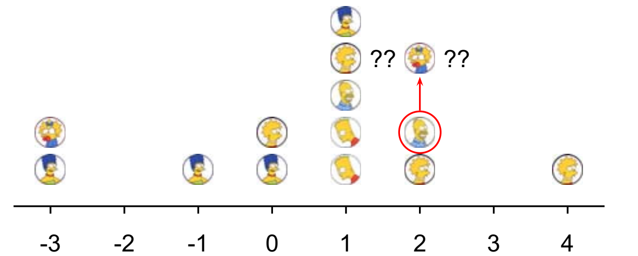

The daughter chooses site 1 with probability $p(2,1)$ or site 3 with probability $p(2,3)$.

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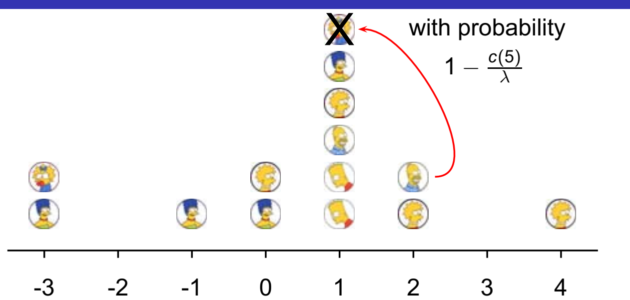

If the daughter chooses site 1, she survives with probability $\frac{c(5)}{\lambda}$ and is eliminated with probability $1 - \frac{c(5)}{\lambda}$.

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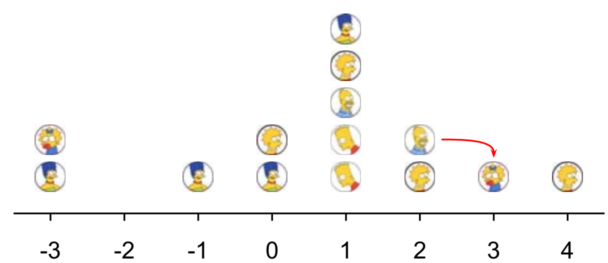

If the daughter chooses site 1, she survives with probability $\frac{c(5)}{\lambda}$ and is eliminated with probability $1 - \frac{c(5)}{\lambda}$.
If she chooses site 3, she survives with probability 1.

Examples of c

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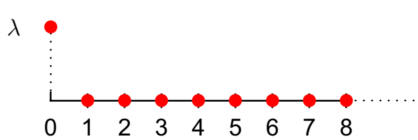
- Contact process: $c(k) = \lambda 1\!\!1_0(k)$.



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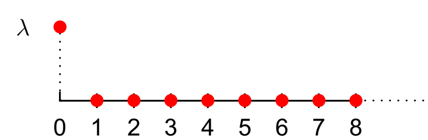
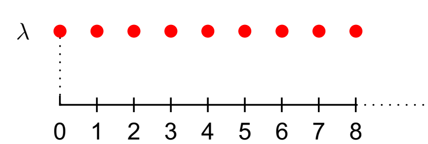
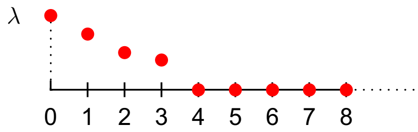


- BRW: $c(k) = \lambda$ for all k.



Examples of *c*

- Here the maximal number of individuals per site is 4.



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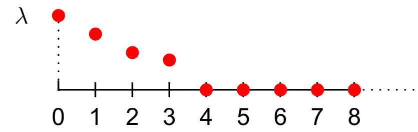
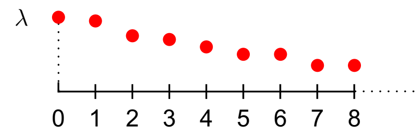


- Here the maximal number of individuals per site is ∞.



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Main issues

1. Construction of the process starting from any (admissible) configuration.

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1. Construction of the process starting from any (admissible) configuration.
2. Study of the phase diagram.

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Main issues

1. Construction of the process starting from any (admissible) configuration.
2. Study of the phase diagram.
3. Construction of an invariant measure for the “ecological equilibrium” phase.

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Existence

The state space $\Omega = \mathbb{N}^X$ is not locally compact: this prevents us from using the classical Hille-Yosida approach.

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The state space $\Omega = \mathbb{N}^X$ is not locally compact: this prevents us from using the classical Hille-Yosida approach.

We use a technique developed by Liggett and Spitzer in 1981 (*Ergodic theorems for coupled random walks and other systems with locally interacting components*, Z. Wahrsch. Verw. Gebiete **56**).

For more details

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Phase diagram

Let $\eta_t(x)$ be the number of individuals living at x at time t.

Extinction phase

If $c(0) \leq 1$ then $\lim_{t\to+\infty} \eta_t(x) = 0$ a.s. for all $x \in X$ and for all finite initial configurations.

Proof: by coupling with a subcritical BRW.

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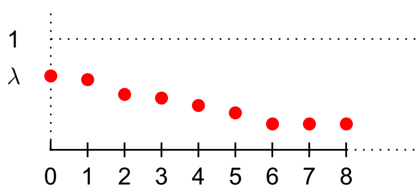

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Phase diagram

Let $c(+\infty) := \lim_{k\to\infty} c(k)$.

Explosion phase

If $c(+\infty) > 1$ then there is survival with positive probability. $\lim_{t\to+\infty} \mathbb{E}[\eta_t(x)] = +\infty$ for all x and for all initial (nonzero) configuration.

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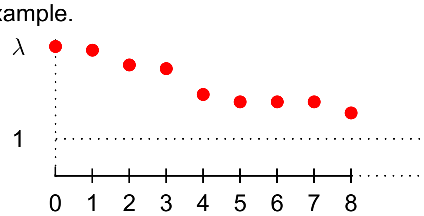

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Ecological equilibrium

The ecological equilibrium phase

If $c(0) > \lambda_{CP}$ and $c(+\infty) < 1$ then there is survival with positive probability.
Moreover, $\limsup_{t\to+\infty} \mathbb{E}[\eta_t(x)] < +\infty$, uniformly for all x and all *bounded* initial configurations.

Ecological equilibrium

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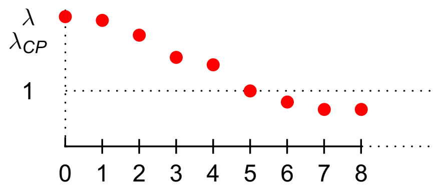

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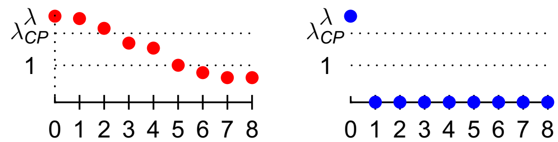
Ecological equilibrium

Ideas for the proof of survival:

- $c(0) > \lambda_{CP}$ ensures survival (via a coupling with the contact process).

Example:



RBRW $\geq$ λ-contact process

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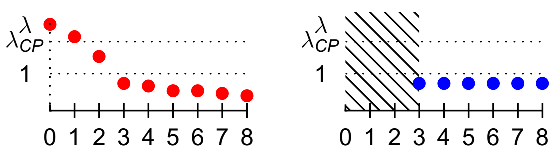
Ecological equilibrium

Ideas for the proof of non-explosion:

- $c(+\infty) < 1$ ensures non-explosive behaviour (via coupling with a BRW with a finite number per site of immortal individuals).

Example:



RBRW $\leq$ BRW with 3 immortal particles per site

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Invariant measure

The ecological equilibrium phase: an invariant measure

If $c(0) > \lambda_{CP}$ and $c(+\infty) < 1$ then there exists a non-trivial invariant measure μ.

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Ideas for the proof:

- truncate c: consider $c_n(k) := c(k)\mathbb{1}_{k \leq n}$;

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Properties of μ

- If the transition matrix P is translation invariant, then so is μ.

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Properties of μ

- If the transition matrix P is translation invariant, then so is μ.
- The expected number of particles per site according to μ is finite.

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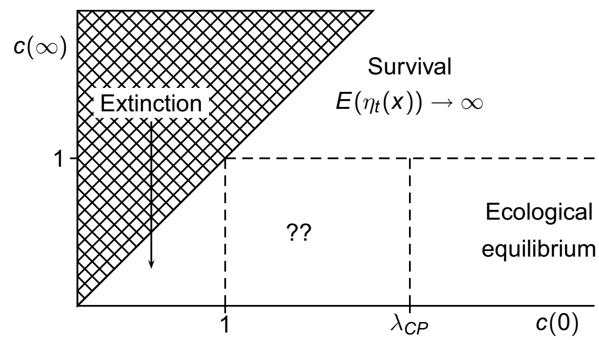

If $c(\infty) < 1$ and $c(0) \in (1, \lambda_{CP})$ we did not characterize the behaviour.

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Open questions

How should we deal with the case $c(\infty) < 1$ and $c(0) \in (1, \lambda_{CP})$?

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How should we deal with the case $c(\infty) < 1$ and $c(0) \in (1, \lambda_{CP})$?

Clearly some of these processes die out and some survive in ecological equilibrium.

We should consider the whole shape of c.

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- Is the invariant measure μ extremal?

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- Can we describe the set of invariant measures?

THANK YOU
FOR YOUR ATTENTION

The construction

1. Define a norm on $\mathbb{N}^X$:
$$\|\eta\| = \sum_x \eta(x)\alpha^{-|x|},$$
the configuration space Ω = finite norm ηs.

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the configuration space Ω = finite norm ηs.
2. Define an operator $\mathcal{L} : \mathrm{Lip}(\Omega) \to \mathbb{R}^\Omega$:
$$(\mathcal{L}f)(\eta) = \sum_x \eta(x) \left[\partial_x^- f(\eta) + \sum_y p(x,y)c(\eta(y))\partial_y^+ f(\eta) \right].$$
$\mathcal{L}$ generates the process starting from any finite configuration.

The construction

3. Choose a sequence of finite subgraphs $\Lambda_n \uparrow X$.

The construction

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   c. μ such that $\mu(\|\eta\|) < \infty$ is invariant $\iff \mu(\mathcal{L}f) = 0$ for all $f \in \mathrm{Lip}(\Omega)$.
7. Extend S_t to a large family of measurable functions f and thus identify a unique Markov process.

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The BRW when $(X, P) \neq (\mathbb{Z}^d, \mathrm{srw})$

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In general there are two parameters, $\rho \neq \theta$:

$$\rho = \limsup_n \sqrt[n]{p^{(n)}(x,y)}, \qquad \theta = \lim_n \sqrt[n]{\sup_x \sum_y p^{(n)}(y,x)}.$$

1. If $\lambda > 1/\rho$ then $\mathbb{E}(\eta_t(x)) \to \infty$ for all initial $\eta \neq \underline{0}$.

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3. If $\lambda < 1/\theta$ then $\limsup_t \mathbb{E}(\eta_t(x)) < \infty$ uniformly for all bounded η.
4. If $\lambda \in (1/\theta, 1/\rho)$ then there are examples (srw on the homogeneous tree) where with an initial finite η the process goes extinct, while with an initial bounded but infinite η, $\mathbb{E}(\eta_t(x)) \to \infty$.

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