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## SEMIMARTINGALE REFLECTING BROWNIAN MOTIONS IN THE ORTHANT

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**Abstract.** This paper surveys recent work on semimartingale reflecting Brownian motions in the orthant. These diffusion processes have been proposed as approximate models of multiclass open queueing networks in heavy traffic. The topics covered by this survey are the problems of existence and uniqueness, recurrence classification and stationary distributions for these diffusions.

**1. Introduction.** Reflecting Brownian motions in the orthant have been proposed as approximate models of open queueing networks in heavy traffic (see Harrison-Nguyen [19]). Such networks are of current interest for studying congestion and delay in computer communication and manufacturing systems. For single class and feedforward multiclass networks with first-in-first-out service discipline, limit theorems to justify the diffusion approximation have been proved by Reiman [37] and Peterson [36], respectively. An outstanding open problem is to prove such a limit theorem for multiclass networks with feedback. Indeed, it has recently been discovered (see for example, Lu-Kumar [29], Rybko-Stolyar [39], Dai-Wang [11], Whitt [44], Bramson [3], [4] and Seidman [40]), that the conditions for stability of a multiclass network are not well understood. There is currently a good deal of activity directed towards resolving this problem which is a precursor to the development of a heavy traffic diffusion limit theorem for such networks. Despite the lack of a limit theorem in the multiclass case, it is still of interest and also of potential benefit for this approximation phase to develop a theory for the diffusions. This article focusses on the latter by surveying recent work on reflecting Brownian motions in the orthant. In fact, attention will be confined here to *semimartingale* reflecting Brownian motions in the orthant (SRBMs). The reasons for this are that (a) many of the queueing networks of interest lead to such processes (although there are some models of routing that lead to non-semimartingale reflecting Brownian motions — see [26] for a two-dimensional example), and (b) there currently is no theory for non-semimartingale reflecting Brownian motions in dimensions higher than two (see [6], [43], [45], [13], [46] for some results on the one and two dimensional cases).

This survey will concentrate on the following two topics for SRBMs:

- (i) Synthesis — existence and uniqueness with given geometric data,
- (ii) Analysis — recurrence classification and stationary distributions.

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**2. Synthesis — Existence and Uniqueness.** This section concerns the problem of existence and uniqueness of semimartingale reflecting Brownian motions in the orthant (SRBMs). Loosely speaking, such a process has a semimartingale decomposition such that in the interior of the orthant it behaves like a Brownian motion with a constant drift and covariance matrix, and at each of the  $(d - 1)$ -dimensional boundary faces, the finite variation part of the process increases in a given direction (constant for a particular face), so as to confine the process to the orthant. For historical reasons, this “pushing” at the boundary is called instantaneous reflection.

For a more precise description of an SRBM, the following notation is needed. Let  $d$  be a positive integer,  $S = \{x \in \mathbb{R}^d : x_i \geq 0 \text{ for } i = 1, \dots, d\}$ ,  $\theta$  be a constant vector in  $\mathbb{R}^d$ ,  $\Gamma$  be a  $d \times d$  non-degenerate covariance matrix (symmetric and positive definite), and  $R$  be a  $d \times d$  matrix. A triple  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$  is called a *filtered space* if  $\Omega$  is a set,  $\mathcal{F}$  is a  $\sigma$ -field of subsets of  $\Omega$ , and  $\{\mathcal{F}_t\} \equiv \{\mathcal{F}_t, t \geq 0\}$  is an increasing family of sub- $\sigma$ -fields of  $\mathcal{F}$ , i.e., a filtration.

**DEFINITION 2.1.** *An SRBM associated with the data  $(S, \theta, \Gamma, R)$  is a continuous,  $\{\mathcal{F}_t\}$ -adapted,  $d$ -dimensional process  $Z$  together with a family of probability measures  $\{P_x, x \in S\}$  defined on some filtered space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$  such that for each  $x \in S$ , under  $P_x$ ,*

$$(2.1) \quad Z(t) = X(t) + RY(t) \in S \quad \text{for all } t \geq 0,$$

where

- (i)  $X$  is a  $d$ -dimensional Brownian motion with drift vector  $\theta$ , covariance matrix  $\Gamma$ ,  $\{X(t) - \theta t, \mathcal{F}_t, t \geq 0\}$  is a martingale, and  $X(0) = x$ ,  $P_x$ -a.s.,
- (ii)  $Y$  is an  $\{\mathcal{F}_t\}$ -adapted,  $d$ -dimensional process such that  $P_x$ -a.s. for  $i = 1, \dots, d$ ,
  - (a)  $Y_i(0) = 0$ ,
  - (b)  $Y_i$  is continuous and non-decreasing,
  - (c)  $Y_i$  can increase only when  $Z$  is on the face  $F_i \equiv \{x \in S : x_i = 0\}$ , i.e.,  $\int_0^t 1_{F_i}(Z(s)) dY_i(s) = Y_i(t)$  for all  $t \geq 0$ .

One approach to constructing an SRBM is to try to solve the following *deterministic Skorokhod* problem for all continuous paths  $x(\cdot)$  in  $\mathbb{R}^d$  that start in  $S$ . This approach is based on the hope that the solutions might be given by a measurable, adapted path-to-path mapping, which could be applied to the paths of a Brownian motion to yield a “strong solution” of the equations defining an SRBM.

In the following,  $\mathbf{C} \equiv C([0, \infty), \mathbb{R}^d) = \{x : [0, \infty) \rightarrow \mathbb{R}^d, x \text{ is continuous}\}$  and  $\mathbf{C}_+ = \{x \in \mathbf{C} : x(0) \in S\}$ . We endow  $\mathbf{C}$  (and hence  $\mathbf{C}_+$ ) with the topology of uniform convergence on compact time intervals and define the associated  $\sigma$ -fields  $\mathcal{M} = \sigma\{x(s) : 0 \leq s < \infty\}$ ,  $\mathcal{M}_t = \sigma\{x(s) : 0 \leq s \leq t\}$  for all  $t \geq 0$ .

**DEFINITION 2.2. (Skorokhod Problem)** Let  $x \in \mathbf{C}_+$ . Then  $(z, y) \in \mathbf{C} \times \mathbf{C}$  solves the Skorokhod problem (SP) for  $x$  (with respect to  $S$  and  $R$ ) if

- (i)  $z(t) = x(t) + Ry(t) \in S$  for all  $t \geq 0$ ,
- (ii)  $y$  is such that for  $i = 1, \dots, d$ ,
  - (a)  $y_i(0) = 0$ ,
  - (b)  $y_i$  is non-decreasing,
  - (c)  $y_i$  can increase only when  $z$  is on  $F_i$ .

Harrison and Reiman [20] considered this problem in connection with single class open queueing networks. The type of  $R$  matrix that arises from such networks is of the form  $R = I - P'$  where  $P$  has zeros on its diagonal and is a transition matrix for a transient Markov chain on  $d$  states (corresponding to the  $d$  nodes in the network). For such an  $R$  matrix, Harrison and Reiman showed the existence of a Lipschitz continuous path-to-path mapping  $\Phi : \mathbf{C}_+ \rightarrow \mathbf{C} \times \mathbf{C}$  which for each  $x \in \mathbf{C}_+$  yields an adapted solution  $(z, y) = \Phi(x)$  of the Skorokhod problem for  $x$ . (Here adapted means that the mapping  $x \rightarrow (z_t, y_t)$  is  $\mathcal{M}_t$ -measurable for each  $t \geq 0$ .) Moreover, this solution is the unique one for  $x$ . Scrutiny of the proof in [20] reveals that the result extends to the case where  $R = I - Q$ , the matrix  $|Q|$  (obtained by replacing each entry in  $Q$  by its absolute value) has spectral radius strictly less than one, and there are no restrictions on the signs of the entries in  $Q$  (see also Dupuis and Ishii [14]). If we apply this result to the paths of a Brownian motion then we obtain a strong solution of the SRBM equations. In summary we have the following.

**THEOREM 2.1.** *Suppose that  $R = I - Q$  where  $|Q|$  has spectral radius strictly less than one. Then for each  $x \in \mathbf{C}_+$ , there is a unique adapted solution  $\Phi(x) = (z, y)$  of the Skorokhod problem for  $x$ . Define  $X, Y, Z$  on  $\mathbf{C}_+$  by  $X(x) = x$ ,  $(Z, Y)(x) = \Phi(x)$  for each  $x \in \mathbf{C}_+$ , and for each  $x_0 \in S$ , let  $P_{x_0}$  denote the probability measure on  $(\mathbf{C}_+, \mathcal{M})$  under which the canonical path  $X = x(\cdot)$  is a Brownian motion with drift  $\theta$ , covariance matrix  $\Gamma$ , and starts from  $x_0$ . Then  $Z$  together with  $\{P_{x_0}, x_0 \in S\}$  is an SRBM associated with  $(S, \theta, \Gamma, R)$ . Furthermore, it defines a Feller continuous strong Markov process.*

Beyond the result of Harrison and Reiman [20], it is relatively easy to see that a necessary condition for one to be able to solve the Skorokhod problem for each  $x \in \mathbf{C}_+$  is that at any point on the boundary of  $S$ , there is a positive linear combination of the vectors of reflection that one is allowed to use there, which points into the interior of the state space. This can also be shown to be necessary for the existence of an SRBM starting from each point in  $S$  (see Theorem 2.3 below). This geometric condition can be expressed succinctly as the following algebraic *completely-S* condition on the reflection matrix  $R$ .

**DEFINITION 2.3.** *A principal submatrix of the  $d \times d$  matrix  $R$  is any square matrix obtained from  $R$  by deleting all rows and columns from  $R$  with indices in some (possibly empty) subset of  $\{1, \dots, d\}$ . The matrix  $R$*

is completely- $\mathcal{S}$  if and only if for each principal submatrix  $\tilde{R}$  of  $R$  there is  $\tilde{x} \geq 0$  such that  $\tilde{R}\tilde{x} > 0$ .

Completely- $\mathcal{S}$  matrices are known in the operations research literature. Alternative names for these matrices are strictly semimonotone or completely- $\mathcal{Q}$  matrices. A useful property is that  $R$  is completely- $\mathcal{S}$  if and only if its transpose  $R'$  is completely- $\mathcal{S}$  (see [38], Lemma 3, p. 91).

If we denote the columns of  $R$  by  $v_1, \dots, v_d$  and the inward unit normals to the faces  $F_i$  by  $n_i$  for  $i = 1, \dots, d$ , then the completely- $\mathcal{S}$  condition can be written in coordinate form as

(S.a) for each  $\mathbf{K} \subset \{1, \dots, d\}$  there is a positive linear combination  $v = \sum_{i \in \mathbf{K}} a_i v_i$  ( $a_i > 0 \forall i \in \mathbf{K}$ ) of the  $\{v_i, i \in \mathbf{K}\}$  such that  $n_i \cdot v > 0$  for all  $i \in \mathbf{K}$ ,

and its adjoint form is:

(S.b) for each  $\mathbf{K} \subset \{1, \dots, d\}$  there is a positive linear combination  $n = \sum_{i \in \mathbf{K}} b_i n_i$  ( $b_i > 0 \forall i \in \mathbf{K}$ ) of the  $\{n_i, i \in \mathbf{K}\}$  such that  $n \cdot v_i > 0$  for all  $i \in \mathbf{K}$ .

Two sets of authors, namely, Bernard and El Kharroubi [2] and Mandelbaum and Van der Heyden [34], independently showed that there is a solution of the Skorokhod problem for all  $x \in C_+$  when  $R$  is completely- $\mathcal{S}$ . Briefly their approach may be described as follows. Approximate the path of  $x \in C_+$  by a piecewise linear path  $\tilde{x}$  and control this approximate path using a pushing path  $\tilde{y}$  which keeps the resultant path  $\tilde{z} = \tilde{x} + R\tilde{y}$  in  $S$ . Control of the affine segments of  $\tilde{x}$  is achieved by solving linear complementarity problems of the following form.

**DEFINITION 2.4. (Linear Complementarity Problem)** Let  $x \in \mathbb{R}^d$ . Then  $(z, y) \in \mathbb{R}^d \times \mathbb{R}^d$  is a solution of the linear complementarity problem for  $x$  and  $R$  if

- (i)  $z = x + Ry \in S$ ,
- (ii) for  $i = 1, \dots, d$ ,
  - (a)  $y_i \geq 0$ ,
  - (b)  $z_i y_i = 0$ .

One can pass to a limit in the approximation (possibly along a subsequence) using the following oscillation estimate (see Lemma 2.1) to obtain the necessary compactness. This then yields a solution of the Skorokhod problem, which is also known as the *dynamic complementarity problem*. For the statement of the oscillation estimate we need the following. For any continuous function  $f$  defined on  $[t_1, t_2]$  into  $\mathbb{R}^k$ , some  $k \geq 1$ , let

$$\text{Osc}(f, [t_1, t_2]) = \sup_{t_1 \leq s \leq t \leq t_2} |f(t) - f(s)|.$$

**LEMMA 2.1.** Suppose that  $R$  is completely- $\mathcal{S}$ . Then there is a constant  $\kappa$  which depends only on  $R$ , such that for any  $x \in C_+$  and solution  $(z, y)$  of the Skorokhod problem for  $x$ , we have for all  $0 \leq t_1 < t_2 < \infty$ ,

$$\text{Osc}(z, [t_1, t_2]) \leq \kappa \text{Osc}(x, [t_1, t_2]), \quad \text{Osc}(y, [t_1, t_2]) \leq \kappa \text{Osc}(x, [t_1, t_2]).$$

*Proof.* See Bernard and El Kharroubi [2] or Dai and Williams [12], Lemma 4.3.  $\square$

Summarizing the above we have the following.

**THEOREM 2.2.** *There is a solution of the Skorokhod problem for each  $x \in C_+$  if and only if  $R$  is completely- $\mathcal{S}$ .*

The linear complementarity problem has unique solutions whenever  $R$  is a  $P$ -matrix (all principal minors are positive) (see [7]). However, solutions of the Skorokhod problem need not be unique, even if  $R$  is a  $P$ -matrix, as shown by examples of Bernard and El Kharroubi [2] and Mandelbaum [33]. This non-uniqueness has the further consequence that the authors of [2] and [34] were not able to show that there is a measurable path-to-path mapping  $\Phi : C_+ \rightarrow C \times C$  which for each  $x \in C_+$  yields an adapted solution  $(z, y) = \Phi(x)$  of the Skorokhod problem for  $x$ . The reason for this is that in taking the limit of the approximate solutions, one may need to pass to a subsequence that depends on  $x$ , and since there is no guarantee of uniqueness, one is not able to show that the solution can be chosen to be an adapted function of  $x$ . Consequently, Theorem 2.2 cannot be used to construct strong solutions for the SRBM equations. It is still an open problem to determine a necessary and sufficient condition for the existence and uniqueness of strong solutions to the SRBM equations.

Taking a different approach, in [41], Taylor and Williams sought weak solutions of the SRBM equations. That is, rather than trying to find  $(Z, Y)$  adapted to a given  $X$ , they sought  $(X, Y, Z)$  together which satisfied the SRBM equations. This approach yielded the following weak existence and uniqueness of an SRBM.

**THEOREM 2.3.** *There exists an SRBM with data  $(S, \theta, \Gamma, R)$  if and only if  $R$  is completely- $\mathcal{S}$ . In this case, the SRBM is unique in law and defines a Feller continuous strong Markov process.*

*Proof of Theorem 2.3.* The necessity of the completely- $\mathcal{S}$  condition follows easily by considering starting points that range over all of the facets of the boundary with dimensions between zero and  $d - 1$  inclusive (see Reiman and Williams [38] for a proof). The sufficiency of the completely- $\mathcal{S}$  condition, as well as the uniqueness in law, is proved in Taylor and Williams [41]. An alternative method for proving existence that exploits Kurtz's [27] patchwork and constrained martingale problem methodology is used in Dai and Williams [12]. The uniqueness proof follows the general line of argument in Bass and Pardoux [1] or Kwon and Williams [28], but hinges on an ergodic property which is verified using the oscillation estimate of Lemma 2.1. The Feller continuity and strong Markov property follow by standard arguments once the existence, uniqueness and tightness of the associated probability measures are established (see [41], p. 316).  $\square$

*Remark.* An extension of the results in this section to semimartingale reflecting Brownian motions in convex polyhedrons has been obtained by Dai and Williams [12]. Their result is sharpest for simple convex polyhedrons. In a simple polyhedron, precisely  $d$  faces meet at a vertex in  $d$  dimensions,

and then a natural generalization of the completely- $\mathcal{S}$  condition (S.a) is necessary and sufficient for the weak existence and uniqueness of an SRBM. For non-simple convex polyhedrons, a generalized completely- $\mathcal{S}$  condition together with an adjoint condition (cf. (S.b)) provides a sufficient condition for the weak existence and uniqueness of an SRBM. See [12] for more details. Reflecting Brownian motions in convex polyhedrons are of interest as approximate models of closed, capacitated and fork-join networks. In particular, the latter can lead to SRBMs in non-simple convex polyhedrons (see Nguyen [35]).

**3. Analysis — Recurrence and Stationary Distributions.** Henceforth we assume that the matrix  $R$  is completely- $\mathcal{S}$  and without loss of generality we assume that it has ones on its diagonal (this can always be achieved by a renormalization of the vectors of reflection). We let  $Z$  together with  $\{P_x, x \in S\}$ , defined on some filtered space, denote a realization of the SRBM associated with  $(S, \theta, \Gamma, R)$ . Expectations under  $P_x$  will be denoted by  $E_x$ .

We first consider the problem of determining conditions for positive recurrence of  $Z$ . If  $Z$  is positive recurrence then there is a unique stationary distribution for  $Z$ . The second part of this section is devoted to the problem of characterizing (and computing) such a stationary distribution.

**DEFINITION 3.1.** *The SRBM  $Z$  is positive recurrent if for each closed set  $A$  in  $S$  having positive Lebesgue measure we have  $E_x[\tau_A] < \infty$  for all  $x \in S$ , where  $\tau_A = \inf\{t \geq 0 : Z(t) \in A\}$ .*

Recently Dupuis and Williams [15] proved that a sufficient condition for positive recurrence of the SRBM  $Z$  is that all solutions of a related deterministic Skorokhod problem are attracted to the origin in the following sense.

**DEFINITION 3.2.** *A path  $z \in \mathbf{C}$  is attracted to the origin if and only if for each  $\epsilon > 0$  there is  $T < \infty$  such that  $|z(t)| \leq \epsilon$  for all  $t \geq T$ .*

**THEOREM 3.1.** *Suppose that all solutions  $z$  of the Skorokhod problem for drift paths  $x$  of the form  $x(t) = x_0 + \theta t$ ,  $t \geq 0$ ,  $x_0 \in S$ , are attracted to the origin. Then the SRBM  $Z$  is positive recurrent and it has a unique stationary distribution.*

*Proof.* The details of the proof can be found in Dupuis and Williams [15]. We briefly mention some of the key aspects here. Firstly, because of the Brownian motion diffusive aspect of  $Z$ , it suffices for the positive recurrence of  $Z$  to show that  $E_x[\tau_r] < \infty$  for all  $x \in S$  and all  $r$  sufficiently large, where  $\tau_r = \inf\{t \geq 0 : |Z(t)| \leq r\}$  (see the proof of Theorem 2.6 in [15]). The mechanism used in [15] to establish the finiteness of these first moment hitting times is to construct a Lyapunov function  $f$  that has the following properties. The function  $f$  is twice continuously differentiable on  $S \setminus \{0\}$  and

- (i) given  $N < \infty$ , there is  $M < \infty$  such that  $x \in S$  and  $|x| \geq M$  imply  $f(x) \geq N$ ,