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Sharp Inequalities for Heat Kernels of Schrödinger Operators and Applications to Spectral Gaps

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This paper derives inequalities for multiple integrals from which sharp inequalities for ratios of heat kernels and integrals of heat kernels of certain Schrödinger operators follow. Such ratio inequalities imply sharp inequalities for spectral gaps. The multiple integral inequalities, although very different, are motivated by the now classical Brascamp–Lieb–Luttinger rearrangement inequalities. © 2000 Academic Press

Contents.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let D be a bounded domain in $\mathbf{R}^n$ and let V be a nonnegative bounded potential in D. It is well known that the eigenvalues of the Dirichlet problem

$$\begin{cases} -\Delta\varphi + V\varphi = \lambda\phi, & \text{in } D \\ \varphi = 0 & \text{on } \partial D \end{cases}$$

are discrete and satisfy $0 < \lambda^V_{1,D} < \lambda^V_{2,D} \leqslant \lambda^V_{3,D} \ldots$. When the potential is identically zero we will just write $\lambda_{i,D}$ for these eigenvalues. The quantity

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$\lambda_{2,D}^V - \lambda_{1,D}^V$ is called the spectral gap. Motivated by problems in mathematical physics concerning the behavior of free Boson gases, M. van den Berg [8] made the following conjecture (see also [1, 2]):

Conjecture 1. Suppose $D \subset \mathbf{R}^n$ is convex of diameter d_D and that V is a nonnegative convex potential in D. Then

$$\lambda_{2,D}^V - \lambda_{1,D}^V > \frac{3\pi^2}{d_D^2}, \tag{1}$$

with the lower bound approached when $V = 0$ and the domain becomes a thin rectangular box.

It is easy to see, by considering a domain in the shape of a barbell, that the convexity is needed even when $V = 0$. The first partial result on this conjecture was obtained by I. M. Singer *et al.* [20] who proved that

$$\frac{\pi^2}{4d_D^2} \leqslant \lambda_{2,D}^V - \lambda_{1,D}^V \leqslant \frac{n\pi^2}{r_D^2} + \frac{4(M-m)}{n}, \tag{2}$$

where $M = \sup_D V$, $m = \inf_D V$, and r_D is the supremum of the radii of all the disks contained in D. This quantity is called the inner radius of D. Their method, based on the maximum principle applied to an appropriately constructed "P-functional" (a la L. Payne) was also used by Yu and Zhong [23] to improve the lower bound in (2) by a factor of 4 and by Ling [16] to obtain the strict inequality

$$\lambda_{2,D}^V - \lambda_{1,D}^V > \frac{\pi^2}{d_D^2}.$$

When the domain D consists of an interval in $\mathbf{R}$ and V is a symmetric single-well potential, M. Ashbaugh and R. Benguria [1, 2] proved the lower bound in (1). The full conjecture for intervals on the real line and for arbitrary nonnegative convex potentials was proved by R. Lavine [13]. The proofs in [1, 2] are based on variational characterizations and appropriately chosen tests functions. The proof in [13] is based on a similar approach and on analysis of the spectrum associated with a family of one-parameter continuous potentials derived from the potential V.

In [21, 22], R. Smits took a different approach. The spectral gap $\lambda_{2,D} - \lambda_{1,D}$ is the first eigenvalue of the operator associated with the Brownian motion conditioned to remain forever in the domain D. In a smooth convex domain the behavior of this diffusion near the boundary is very similar to the behavior of the reflected Brownian motion as can be seen from the gradient estimates for the first Dirichlet eigenfunction proved in R. Bañuelos and M. Pang [5]. Motivated by this, and in part also by the study of *intrinsic ultracontractivity* and rates to equilibrium for the Brownian motion conditioned to remain forever in D, Smits [21] was able to modify the classical proof of Payne and Weinberger [3, p. 155] for the lower bound of the first nontrivial Neumann eigenvalue in terms of the diameter of the domain to give a new proof of Ling's lower bound. His arguments also improves the upper bound in (2). For the connection to *intrinsic ultracontractivity* and rates to equilibrium, as well as a reformulation of Conjecture 1 in terms of these quantities, we refer the reader to Bañuelos [4].

The purpose of this paper is to prove various ratio inequalities for heat kernels and integrals of heat kernels from which sharp spectral gap inequalities immediately follow. We believe these ratio inequalities and their proofs are of independent interest. To do this we restrict ourselves to a class of domains which are symmetric with respect to both axes as well as convex in both axes. These domains, however, do not have to be convex.

Let D be a bounded domain in $\mathbf{R}^2$ which is symmetric with respect to the y-axis. We say that D is convex in x (respectively y) if every line parallel to the x-axis (respectively the y-axis) which intersect D cuts ∂D in at most two points. Note that a domain can be convex in x and y without being convex. Let C be the smallest rectangle containing D. Translating D, if necessary, we can suppose that

$$C=(-b, b)\times(-a, a).$$

Set

$$D^+=D\cap\{(x, y)\in\mathbf{R}^2 : x>0\}, \qquad C^+=(0, b)\times(-a, a).$$

We use B_s to denote two dimensional Brownian motion and for any Borel set A we will write τ_A for the first exit time of the Brownian motion from A. For $0<x, \tilde{x}<b$ and $y\in(-a, a)$ consider

$$\frac{P_{(x, y)}(\tau_{C^+}>t)}{P_{(\tilde{x}, y)}(\tau_C>t)}.$$

By independence we have

$$P_{(x, y)}(\tau_{C^+}>t)=P_x(\tau_{(0, b)}>t)\, P_y(\tau_{(-a, a)}>t),$$

and

$$P_{(\tilde{x}, y)}(\tau_C>t)=P_{\tilde{x}}(\tau_{(-b, b)}>t)\, P_y(\tau_{(-a, a)}>t).$$

Thus

$$\frac{P_{(x,\,y)}(\tau_{C^+} > t)}{P_{(\tilde{x},\,y)}(\tau_C > t)} = \frac{P_x(\tau_{(0,\,b)} > t)}{P_{\tilde{x}}(\tau_{(-b,\,b)} > t)}.$$

For the rest of the paper we will assume that $V(x, y)$ is a positive bounded, continuous potential which is symmetric in the y-axis and increasing as a function of x. That is,

$$V(x, y) = V(-x, y),$$

for all $(x, y) \in \mathbf{R}^2$ and $V(\cdot, y)$ is increasing in $\mathbf{R}^+$ for y fixed. Let

$$A_+ = \{(x, y) \in \mathbf{R}^2 : y \geqslant 0,\ y \leqslant x\},$$

and

$$A_- = \{(x, y) \in \mathbf{R}^2 : y \geqslant 0,\ y \leqslant -x\}.$$

For $z = (x, y) \in A_+ \cup A_-$, define

$$\hat{z} = \begin{cases} z^+ = (y, x) & \text{if} \quad y \leqslant x \\ z^- = (-y, -x) & \text{if} \quad y \leqslant -x. \end{cases}$$

Note that z^+ and z^- are the reflections of z with respect to the lines $y = x$ and $y = -x$, respectively. With this notation we have

THEOREM 1. *Suppose D is a bounded domain in $\mathbf{R}^2$ which is symmetric with respect to the y-axis and convex in x. Let D^+, C, C^+ and V be as above. Suppose $(w_1, v_1) \in D^+$, $(\tilde{w}_1, v_1) \in D$, $(w_2, v_2) \in C^+$, and $(\tilde{w}_2, v_2) \in C$ are such that*

- *if $z = (\tilde{w}_2, w_1) \in A_+ \cup A_-$, then $\hat{z} = (\tilde{w}_1, w_2)$, and*
- *if $z = (\tilde{w}_2, w_1) \notin A_+ \cup A_-$, then $\tilde{w}_2 = \tilde{w}_1$ and $w_2 = w_1$.*

Then for all $t > 0$,

$$\frac{E_{(w_1,\,v_1)}\{\exp(-\int_0^t V(B_s)\,ds),\ \tau_{D^+} > t\}}{E_{(\tilde{w}_1,\,v_1)}\{\exp(-\int_0^t V(B_s)\,ds),\ \tau_D > t\}} \leqslant \frac{P_{(w_2,\,v_2)}(\tau_{C^+} > t)}{P_{(\tilde{w}_2,\,v_2)}(\tau_C > t)} = \frac{P_{w_2}(\tau_{(0,\,b)} > t)}{P_{\tilde{w}_2}(\tau_{(-b,\,b)} > t)}. \tag{3}$$

In particular for all $(x, 0) \in D^+$ and all $t > 0$,

$$\frac{E_{(x,\,0)}\{\exp(-\int_0^t V(B_s)\,ds),\ \tau_{D^+} > t\}}{E_{(x,\,0)}\{\exp(-\int_0^t V(B_s)\,ds),\ \tau_D > t\}} \leqslant \frac{P_x(\tau_{(0,\,b)} > t)}{P_x(\tau_{(-b,\,b)} > t)}. \tag{4}$$

The motivation for these results arises from probabilistic considerations but, of course, both inequalities (3) and (4) can be stated without reference to Brownian motion in terms of integrals of heat kernels. More precisely, let $P_V^D(t, z, w)$ and $P_V^{D^+}(t, z, w)$ be the Dirichlet heat kernels of the operator $-\frac{1}{2}\Delta + V$ in D and D^+, respectively. Denote the Dirichlet heat kernel for the intervals $(-b, b)$ and $(0, b)$ with zero potentials by $P^{(-b, b)}(t, x, y)$ and $P^{(0, b)}(t, x, y)$, respectively. Then (4) is equivalent to

$$\frac{\int_{D^+} P_V^{D^+}(t, (x, 0), w)\, dw}{\int_D P_V^D(t, (x, 0), w)\, dw} \leqslant \frac{\int_0^b P^{(0, b)}(t, x, y)\, dy}{\int_{-b}^b P^{(-b, b)}(t, x, y)\, dy},$$

for all $(x, 0) \in D^+$ and all $t > 0$. Inequality (3) has a similar formulation.

The following corollaries, as we shall see below, are immediate consequences of the inequality (4).

Corollary 1. *Suppose $V = 0$ and let D be a bounded domain in $\mathbf{R}^2$ which is symmetric with respect to both coordinate axes and convex in both axes. Let $l = 2b$ be the length of its major axis and set $I = (-b, b)$. Then*

$$\lambda_{2, D} - \lambda_{1, D} \geqslant \lambda_{2, I} - \lambda_{1, I} = \frac{3\pi^2}{l^2}.$$

Corollary 2. *Let $I = (-b, b)$ and let V be a nonnegative potential which satisfies $V(x) = V(-x)$ and is increasing on $(0, b)$. Then*

$$\lambda^V_{2, I} - \lambda^V_{1, I} \geqslant \lambda_{2, I} - \lambda_{1, I} = \frac{3\pi^2}{l^2}.$$

In [9], B. Davis proves inequalities for ratios of heat kernels (not

The paper is organized as follows. In Section 2, we prove an inequality for multiple integrals. This inequality, although very different, is inspired by the inequalities of H. J. Brascamp *et al.* [7]. This inequality also has consequences for other Markov processes whose transition probabilities are radially symmetric and decreasing such as the symmetric stable processes. (See Corollaries 3 and 4 below.) In Section 3, we use this inequality to prove Theorem 1 and in Section 4, we show how Corollaries 1 and 2 follow from (3) or (4). This section also contains a result (Corollary 6) concerning the location of the nodal line and the multiplicity of the second eigenfunction for symmetric convex domains which are "long enough." In Section 5 we present an example of a potential in a disk which satisfies the hypothesis of Theorem 1 for which the nodal line is a circle and the first Dirichlet eigenvalue of D^+ is strictly larger than the second Dirichlet eigenvalue of D. We end with Section 6 which contains a version of Theorem 1 for ratios of heat kernels. These results follow from inequalities on multiple integrals and the representation of the heat kernel in terms of the Brownian bridge.

2. AN INEQUALITY FOR MULTIPLE INTEGRALS

In this section we derive an inequality for multiple integrals of radially symmetric decreasing functions from which we will obtain Theorem 1. This inequality has other interesting consequences as we shall see below. We begin with the following simple geometric lemma which is fundamental in the proof of Theorem 2 below. The quantities A_+, A_-, and $\hat{z}$ are as defined in the introduction.

LEMMA 1. *If $z_1 \in A_+$ and $z_2 \in A_-$, then*

$$|\hat{z}_1 - \hat{z}_2| \leqslant |z_1 - \hat{z}_2| \leqslant |z_1 - z_2|,$$

and

$$|\hat{z}_1 - \hat{z}_2| \leqslant |\hat{z}_1 - z_2| \leqslant |z_1 - z_2|.$$

If $z_1, z_2 \in A_+$ or $z_1, z_2 \in A_-$, then

$$|\hat{z}_1 - \hat{z}_2| = |z_1 - z_2| \leqslant |z_1 - \hat{z}_2| = |\hat{z}_1 - z_2|.$$

Proof. We will denote the dot product of two points z and w by $z \cdot w$. Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. Suppose that $x_1 \geqslant y_1 \geqslant 0$ and $-x_2 \geqslant y_2 \geqslant 0$. Then

$$\begin{aligned} -x_2(x_1-y_1)\geqslant y_2(y_1-x_1) &\Leftrightarrow -x_1x_2-y_1y_2\geqslant -y_1x_2-x_1y_2\\ &\Leftrightarrow -z_1\cdot z_2\geqslant -z_1^+\cdot z_2\\ &\Leftrightarrow |z_1-z_2|^2\geqslant |z_1^+-z_2|^2, \qquad (6)\\ -x_2(x_1+y_1)\geqslant y_2(x_1+y_1) &\Leftrightarrow -y_1x_2-x_1y_2\geqslant y_1y_2+x_1x_2\\ &\Leftrightarrow -z_1^+\cdot z_2\geqslant -z_1^+\cdot z_2^-\\ &\Leftrightarrow |z_1^+-z_2|^2\geqslant |z_1^+-z_2^-|^2. \qquad (7) \end{aligned}$$

On the other hand, we easily see that

$$\begin{aligned} |z_1-z_2^-|\geqslant |z_1^+-z_2^-| &\Leftrightarrow x_1y_2+y_1x_2\geqslant y_1y_2+x_1x_2,\\ |z_1-z_2|\geqslant |z_1-z_2^-| &\Leftrightarrow -x_1x_2-y_1y_2\geqslant x_1y_2+y_1x_2, \end{aligned}$$

which are equivalent to (6) and (7), respectively. Suppose now that $x_1\geqslant y_1$ and $x_2\geqslant y_2$. Then

$$\begin{aligned} x_1(x_2-y_2)\geqslant y_1(x_2-y_2) &\Leftrightarrow -x_1y_2-y_1x_2\geqslant -x_1x_2-y_1y_2\\ &\Leftrightarrow |z_1-z_2^+|\geqslant |z_1-z_2|. \end{aligned}$$

In addition, it is clear that

$$|z_1-z_2|=|z_1^+-z_2^+|, \qquad |z_1^+-z_2|=|z_1-z_2^+|.$$

Note that the case z_1, $z_2\in A_-$ follows from the last two identities by a reflection with respect to the y-axis. This completes the proof. ∎

We now fix $\beta>0$ and define for any $0<\alpha<\beta$,

$$C(\alpha):=(-\beta,\beta)\times(0,\alpha), \qquad \text{and} \qquad \tilde{C}(\alpha):=(-\alpha,\alpha)\times(0,\beta).$$

THEOREM 2. *Let $\alpha_1, ..., \alpha_m$ be such that $0<\alpha_i<\beta$ for $1\leqslant i\leqslant m$ and $p_i(z)$ be radially symmetric nonincreasing functions in $\mathbf{R}^2$ for $1\leqslant i\leqslant m$. Let $V_i(x,y)$, $\tilde{V}_i(x,y)$, $1\leqslant i\leqslant m$, be positive real functions with the property that*

- *$V_i(x,y)=V_i(-x,y)$ and $\tilde{V}_i(x,y)=\tilde{V}_i(-x,y)$, for all $(x,y)\in\mathbf{R}^2$,*
- *$V_i(z)=\tilde{V}_i(\hat{z})$ and $\tilde{V}_i(z)=V_i(\hat{z})$ for all $z\in A_+\cup A_-$,*
- *$V_i(\hat{z})\geqslant\tilde{V}_i(\hat{z})$ for all $z\in A_+\cup A_-$.*

Define the functions

$$\Upsilon_m(z_0)=\int_{C(\alpha_1)}\cdots\int_{C(\alpha_m)}\prod_{i=1}^{m}\left[p_i(z_i-z_{i-1})\,e^{-V_i(z_i)}\right]dz_1\cdots dz_m,$$

and

$$\tilde{\Upsilon}_m(z_0)=\int_{\tilde{C}(\alpha_1)}\cdots\int_{\tilde{C}(\alpha_m)}\prod_{i=1}^{m}\left[p_i(z_i-z_{i-1})\,e^{-\tilde{V}_i(z_i)}\right]dz_1\cdots dz_m.$$

Then for $z_0\in A_+\cup A_-$,

$$\Upsilon_m(z_0)\leqslant\tilde{\Upsilon}_m(\hat{z}_0), \tag{8}$$

$$\Upsilon_m(\hat{z}_0)\leqslant\tilde{\Upsilon}_m(\hat{z}_0), \tag{9}$$

and

$$\Upsilon_m(z_0)+\Upsilon_m(\hat{z}_0)\leqslant\tilde{\Upsilon}_m(z_0)+\tilde{\Upsilon}_m(\hat{z}_0). \tag{10}$$

Let us recall that a nonnegative radially symmetric nonincreasing function h can be written in the form

$$h(x)=\int_0^\infty I_{B(0,\,r)}(x)\,d\mu(r),$$

where μ is a measure on $(0,\infty]$. Therefore we may assume that $p_i=I_{B(0,\,r_i)}$ for some $r_i>0$. We shall now proceed by induction on m. The next lemma is the case of $m=1$.

LEMMA 2. *Fix* $z_0\in A_+\cup A_-$. *Then*

$$\int_{B(z_0,\,r)\cap C(\alpha_1)}e^{-V_1(z_1)}\,dz_1\leqslant\int_{B(\hat{z}_0,\,r)\cap\tilde{C}(\alpha_1)}e^{-\tilde{V}_1(z_1)}\,dz_1, \tag{11}$$

$$\int_{B(\hat{z}_0,\,r)\cap C(\alpha_1)}e^{-V_1(z_1)}\,dz_1\leqslant\int_{B(\hat{z}_0,\,r)\cap\tilde{C}(\alpha_1)}e^{-\tilde{V}_1(z_1)}\,dz_1, \tag{12}$$

and

$$\int_{B(z_0,\,r)\cap C(\alpha_1)}e^{-V_1(z_1)}\,dz_1+\int_{B(\hat{z}_0,\,r)\cap C(\alpha_1)}e^{-V_1(z_1)}\,dz_1$$
$$\leqslant\int_{B(z_0,\,r)\cap\tilde{C}(\alpha_1)}e^{-\tilde{V}_1(z_1)}\,dz_1+\int_{B(\hat{z}_0,\,r)\cap\tilde{C}(\alpha_1)}e^{-\tilde{V}_1(z_1)}\,dz_1, \tag{13}$$

for every $r>0$.

Proof. Since $\tilde{C}(\alpha_1)$ and $C(\alpha_1)$ are symmetric with respect to the y-axis, $V_1(x,y)=V_1(-x,y)$ and $\tilde{V}_1(x,y)=\tilde{V}_1(-x,y)$, it is enough to prove the lemma for $z_0 \in A_+$. Consider the functions $G(z_1) = e^{-V_1(z_1)}$, and $\tilde{G}(z_1) = e^{-\tilde{V}_1(z_1)}$. Let $z_1 \in A_+ \cup A_-$. Since $V_1(z_1) = \tilde{V}_1(\hat{z}_1)$ and $\tilde{V}_1(\hat{z}_1) \leqslant V_1(\hat{z}_1)$, we have that

$$G(z_1) = \tilde{G}(\hat{z}_1) \qquad \text{and} \qquad G(\hat{z}_1) \leqslant \tilde{G}(\hat{z}_1). \tag{14}$$

On the other hand, $\tilde{V}_1(z_1) = V_1(\hat{z}_1)$ and hence

$$G(z_1) + G(\hat{z}_1) = \tilde{G}(z_1) + \tilde{G}(\hat{z}_1). \tag{15}$$

Consider the six regions

$$\begin{aligned} A_{11} &= (\alpha_1, \beta) \times (0, \alpha_1), & \tilde{A}_{11} &= (0, \alpha_1) \times (\alpha_1, \beta), \\ A_{12} &= (-\beta, -\alpha_1) \times (0, \alpha_1), & \tilde{A}_{12} &= (-\alpha_1, 0) \times (\alpha_1, \beta), \\ A_{13} &= (0, \alpha_1) \times (0, \alpha_1), & A_{14} &= (-\alpha_1, 0) \times (0, \alpha_1). \end{aligned}$$

It is clear that $C(\alpha_1) = A_{11} \cup A_{12} \cup A_{13} \cup A_{14}$ and $\tilde{C}(\alpha_1) = \tilde{A}_{11} \cup \tilde{A}_{12} \cup A_{13} \cup A_{14}$. We note that the reflection of A_{11} with respect to the line $x = y$ is $\tilde{A}_{11}$ and that the reflection of A_{12} with respect to the line $y = -x$ is $\tilde{A}_{12}$. We now begin the proof of (11) breaking it into several cases according to this decomposition of the regions $C(\alpha_1)$ and $\tilde{C}(\alpha_1)$. Let $p_1(z) = I_{B(0, r)}(z)$.

Case (11-i). Let $z_1 \in A_{11}$. From Lemma 1 and (14) we know that $G(z_1) = \tilde{G}(z_1^+)$ and that $p_1(z_1 - z_0) = p_1(z_1^+ - z_0^+)$. Thus

$$\int_{A_{11}} p_1(z_1 - z_0)\, G(z_1)\, dz_1 = \int_{A_{11}} p_1(z_1^+ - z_0^+)\, \tilde{G}(z_1^+)\, dz_1.$$

Making the substitution $w_1 = z_1^+$ in the right hand side we obtain

$$\int_{A_{11}} p_1(z_1 - z_0)\, G(z_1)\, dz_1 = \int_{\tilde{A}_{11}} p_1(w_1 - z_0^+)\, \tilde{G}(w_1)\, dw_1.$$

Case (11-ii). Let $z_1 \in A_{12}$. Again by Lemma 1 and (14) we have $G(z_1) = \tilde{G}(z_1^-)$ and $p_1(z_1 - z_0) \leqslant p_1(z_1^- - z_0^+)$. Integrating over A_{12} and making the substitution $w_1 = z_1^-$ in the second integral we obtain

$$\begin{aligned} \int_{A_{12}} p_1(z_1 - z_0)\, G(z_1)\, dz_1 &\leqslant \int_{A_{12}} p_1(z_1^- - z_0^+)\, \tilde{G}(z_1^-)\, dz_1 \\ &= \int_{\tilde{A}_{12}} p_1(w_1 - z_0^+)\, \tilde{G}(w_1)\, dw_1. \end{aligned}$$

Case (11-iii). Let $z_1 \in A_{13} \cap A_+$. We claim that

$$p_1(z_1 - z_0)\, G(z_1) + p_1(z_1^+ - z_0)\, G(z_1^+) \leqslant p_1(z_1 - z_0^+)\, \tilde{G}(z_1) + p_1(z_1^+ - z_0^+)\, \tilde{G}(z_1^+). \tag{16}$$

Since p_1 is an indicator function, $p_1 = 1$ or $p_1 = 0$, and by Lemma 1,

$$p_1(z_1^+ - z_0) = p_1(z_1 - z_0^+) \leqslant p_1(z_1 - z_0) = p_1(z_1^+ - z_0^+).$$

Thus it suffices to consider two cases:

- If $p_1(z_1^+ - z_0) = 1$, then $p_1(z_1 - z_0) = 1$ and (16) follows from (15).
- If $p_1(z_1 - z_0) = 1$ and $p_1(z_1^+ - z_0) = 0$, then (16) follows from (14).

Integrating (16) gives

$$\int_{A_{13} \cap A_+} p_1(z_1 - z_0)\, G(z_1)\, dz_1 + \int_{A_{13} \cap A_+} p_1(z_1^+ - z_0)\, G(z_1^+)\, dz_1 \leqslant \int_{A_{13} \cap A_+} p_1(z_1 - z_0^+)\, \tilde{G}(z_1)\, dz_1 + \int_{A_{13} \cap A_+} p_1(z_1^+ - z_0^+)\, \tilde{G}(z_1^+)\, dz_1.$$

Since

$$\int_{A_{13} \cap A_+} p_1(z_1^+ - z_0)\, G(z_1^+)\, dz_1 = \int_{A_{13} \setminus A_+} p_1(z_1 - z_0)\, G(z_1)\, dz_1,$$

and

$$\int_{A_{13} \cap A_+} p_1(z_1^+ - z_0^+)\, \tilde{G}(z_1^+)\, dz_1 = \int_{A_{13} \setminus A_+} p_1(z_1 - z_0^+)\, \tilde{G}(z_1)\, dz_1,$$

we conclude that

$$\int_{A_{13}} p_1(z_1 - z_0)\, G(z_1)\, dz_1 \leqslant \int_{A_{13}} p_1(z_1 - z_0^+)\, \tilde{G}(z_1)\, dz_1.$$

Case (11-iv). Let $z_1 \in A_{14} \cap A_-$. We claim that

$$p_1(z_1 - z_0)\, G(z_1) + p_1(z_1^- - z_0)\, G(z_1^-) \leqslant p_1(z_1 - z_0^+)\, \tilde{G}(z_1) + p_1(z_1^- - z_0^+)\, \tilde{G}(z_1^t). \tag{17}$$

By Lemma 1,

$$p_1(z_1 - z_0) \leqslant p_1(z_1^- - z_0) \leqslant p_1(z_1^- - z_0^+),$$

and

$$p_1(z_1 - z_0) \leqslant p_1(z_1 - z_0^+) \leqslant p_1(z_1^- - z_0^+).$$

As before, we have two cases to consider.

- If $p_1(z_1 - z_0) = 1$, then $p_1(z_1^- - z_0) = p_1(z_1 - z_0^+) = p_1(z_1^- - z_0^+) = 1$, and (17) follows from (15).
- If $p_1(z_1 - z_0) = 0$ and $p_1(z_1^- - z_0) = 1$, then (17) follows from (14).

Integrating (17) over $A_{14} \cap A_-$ and repeating the argument of the previous case we conclude that

$$\int_{A_{14}} p_1(z_1 - z_0)\, G(z_1)\, dz_1 \leqslant \int_{A_{14}} p_1(z_1 - z_0^+)\, \tilde{G}(z_1)\, dz_1.$$

Putting Cases (11-i)–(11-iv) together, we see that (11) follows.

The proof of (12) is very similar. We again break the region of integration into pieces and as before consider four cases.

Case (12-i). Let $z_1 \in A_{11}$. By (14) and Lemma 1 we know that

$$G(z_1) = \tilde{G}(z_1^+), \qquad p_1(z_1 - z_0^+) \leqslant p_1(z_1^+ - z_0^+).$$

Integrating over A_{11} we find that

$$\int_{A_{11}} p_1(z_1 - z_0^+)\, G(z_1)\, dz_1 \leqslant \int_{\tilde{A}_{11}} p_1(z_1 - z_0^+)\, \tilde{G}(z_1)\, dz_1.$$

Case (12-ii). Let $z_1 \in A_{12}$. By (14) and Lemma 1 we know that

$$G(z_1) = \tilde{G}(z_1^-), \qquad p_1(z_1 - z_0^+) \leqslant p_1(z_1^- - z_0^+).$$

Integrating over A_{12} we have that

$$\int_{A_{12}} p_1(z_1 - z_0^+)\, G(z_1)\, dz_1 \leqslant \int_{\tilde{A}_{12}} p_1(z_1 - z_0^+)\, \tilde{G}(z_1)\, dz_1.$$

Case (12-iii). Let $z_1 \in A_{13} \cap A_+$. We claim that

$$\begin{aligned} &p_1(z_1 - z_0^+)\, G(z_1) + p_1(z_1^+ - z_0^+)\, G(z_1^+) \\ &\quad \leqslant p_1(z_1 - z_0^+)\, \tilde{G}(z_1) + p_1(z_1^+ - z_0^+)\, \tilde{G}(z_1^+). \end{aligned} \tag{18}$$

Recall that

$$p_1(z_1 - z_0^+) \leqslant p_1(z_1^+ - z_0^+).$$

Thus it suffices to consider two cases.

- If $p_1(z_1 - z_0^+) = 1$, then $p_1(z_1^+ - z_0^+) = 1$ and (18) follows from (15).
- If $p_1(z_1^+ - z_0^+) = 1$ and $p_1(z_1 - z_0^+) = 0$, then (18) follows from (14).

Arguing as in Case (11-iii) we conclude that

$$\int_{A_{13}} p_1(z_1 - z_0^+)\, G(z_1)\, dz_1 \leqslant \int_{A_{13}} p_1(z_1 - z_0^+)\, \tilde{G}(z_1)\, dz_1.$$

Case (12-iv). Let $z_1 \in A_{14} \cap A_-$. We claim that

$$\begin{aligned} &p_1(z_1 - z_0^+)\, G(z_1) + p_1(z_1^- - z_0^+)\, G(z_1^-) \\ &\qquad \leqslant p_1(z_1 - z_0^+)\, \tilde{G}(z_1) + p_1(z_1^- - z_0^+)\, \tilde{G}(z_1^-). \end{aligned} \tag{19}$$

By Lemma 1,

$$p_1(z_1 - z_0^+) \leqslant p_1(z_1^- - z_0^+),$$

and again two cases arise.

- If $p_1(z_1 - z_0^+) = 1$, then (19) follows from (15).
- If $p_1(z_1^- - z_0^+) = 1$ and $p_1(z_1 - z_0^+) = 0$, then (19) follows from (14).

We conclude that

$$\int_{A_{14}} p_1(z_1 - z_0^+)\, G(z_1)\, dz_1 \leqslant \int_{A_{14}} p_1(z_1 - z_0^+)\, \tilde{G}(z_1)\, dz_1.$$

Cases (12-i)–(12-iv) prove (12).

It remains to prove (13). The proof is similar.

Case (13-i). Let $z_1 \in A_{11}$. From Lemma 1,

$$p_1(z_1 - z_0) + p_1(z_1 - z_0^+) = p_1(z_1^+ - z_0) + p_1(z_1^+ - z_0^+).$$

Since $G(z_1) = \tilde{G}(z_1^+)$, we have that

$$\begin{aligned} &\int_{A_{11}} [\,p_1(z_1 - z_0) + p_1(z_1 - z_0^+)]\, G(z_1)\, dz_1 \\ &\qquad = \int_{\tilde{A}_{11}} [\,p_1(z_1 - z_0) + p_1(z_1 - z_0^+)]\, \tilde{G}(z_1)\, dz_1. \end{aligned}$$

Case (13-ii). Let $z_1 \in A_{12}$. By Lemma 1,

$$p_1(z_1 - z_0^+) + p_1(z_1 - z_0) \leqslant p_1(z_1^- - z_0^+) + p_1(z_1^- - z_0).$$

From (14) we conclude that

$$\int_{A_{12}} [p_1(z_1 - z_0^+) + p_1(z_1 - z_0)]\, G(z_1)\, dz_1$$

$$= \int_{\tilde{A}_{12}} [p_1(z_1 - z_0^+) + p_1(z_1 - z_0)]\, \tilde{G}(z_1)\, dz_1.$$

Case (13-iii). Let $z_1 \in A_{13} \cap A_+$. We claim that

$$\begin{aligned}
&[p_1(z_1 - z_0) + p_1(z_1 - z_0^+)]\, G(z_1) + [p_1(z_1^+ - z_0) + p_1(z_1^+ - z_0^+)]\, G(z_1^+) \\
&\quad \leqslant [p_1(z_1 - z_0) + p_1(z_1 - z_0^+)]\, \tilde{G}(z_1) \\
&\qquad + [p_1(z_1^+ - z_0) + p_1(z_1^+ - z_0^+)]\, \tilde{G}(z_1^+). \qquad (20)
\end{aligned}$$

It follows from Lemma 1 that

$$p_1(z_1^+ - z_0) + p_1(z_1^+ - z_0^+) = p_1(z_1 - z_0^+) + p_1(z_1 - z_0).$$

Thus (20) follows from (15) and integrating over $A_{13} \cap A_+$ we conclude that

$$\int_{A_{13}} [p_1(z_1 - z_0^+) + p_1(z_1 - z_0)]\, G(z_1)\, dz_1$$

$$\leqslant \int_{A_{13}} [p_1(z_1 - z_0^+) + p_1(z_1 - z_0)]\, \tilde{G}(z_1)\, dz_1.$$

Case (13-iv). Let $z_1 \in A_{14} \cap A_-$. We claim that

$$\begin{aligned}
&[p_1(z_1 - z_0) + p_1(z_1 - z_0^+)]\, G(z_1) + [p_1(z_1^- - z_0) + p_1(z_1^- - z_0^+)]\, G(z_1^-) \\
&\quad \leqslant [p_1(z_1 - z_0) + p_1(z_1 - z_0^+)]\, \tilde{G}(z_1) \\
&\qquad + [p_1(z_1^- - z_0) + p_1(z_1^- - z_0^+)]\, \tilde{G}(z_1^-). \qquad (21)
\end{aligned}$$

By Lemma 1,

$$p_1(z_1 - z_0) \leqslant p_1(z_1^- - z_0) \leqslant p_1(z_1^- - z_0^+),$$

and

$$p_1(z_1 - z_0) \leqslant p_1(z_1 - z_0^+) \leqslant p_1(z_1^- - z_0^+),$$

and we have the corresponding cases.

• If $p_1(z_1 - z_0) = 1$, then (21) follows from (15).

• If $p_1(z_1 - z_0) = 0$ and $p_1(z_1^- - z_0) = p_1(z_1 - z_0^+) = p_1(z_1^- - z_0^+) = 1$, then (21) follows from (14) and (15).

• If $p_1(z_1 - z_0) = p_1(z_1^- - z_0) = 0$ and $p_1(z_1 - z_0^+) = 1$, then (21) follows from (15).

• If $p_1(z_1 - z_0) = p_1(z_1 - z_0^+) = 0$, then (21) follows from (14).

Therefore

$$\int_{A_{14}} [p_1(z_1 - z_0^+) + p_1(z_1 - z_0)]\, G(z_1)\, dz_1$$

$$\leqslant \int_{A_{14}} [p_1(z_1 - z_0^+) + p_1(z_1 - z_0)]\, \tilde{G}(z_1)\, dz_1.$$

It is clear that putting Cases (13-i)–(13-iv) together proves (13) and completes the proof of the Lemma 2. ▮

Proof of Theorem 2. We proceed by induction. Lemma 1 is the case $m = 1$. Now suppose that the result is true for $m - 1$ and let

$$h(z_1) = \int_{C(\alpha_2)} \cdots \int_{C(\alpha_m)} \prod_{i=2}^{m} [p_i(z_i - z_{i-1})\, e^{-V_i(z_i)}]\, dz_2 \cdots dz_m,$$

$$\tilde{h}(z_1) = \int_{\tilde{C}(\alpha_2)} \cdots \int_{\tilde{C}(\alpha_m)} \prod_{i=2}^{m} [p_i(z_i - z_{i-1})\, e^{-\tilde{V}_i(z_i)}]\, dz_2 \cdots dz_m.$$

Consider the functions

$$G(z_1) = h(z_1)\, e^{-V_1(z_1)} \qquad \text{and} \qquad \tilde{G}(z_1) = \tilde{h}(z_1)\, e^{-\tilde{V}_1(z_1)},$$

and let z_1 in $A_+ \cup A_-$. Then $h(z_1) \leqslant \tilde{h}(\hat{z}_1)$, $e^{-V_1(z_1)} = e^{-\tilde{V}_1(\hat{z}_1)}$, $h(\hat{z}_1) \leqslant \tilde{h}(\hat{z}_1)$ and $e^{-V_1(\hat{z}_1)} \leqslant e^{-\tilde{V}_1(\hat{z}_1)}$. Therefore

$$G(z_1) \leqslant \tilde{G}(\hat{z}_1) \qquad \text{and} \qquad G(\hat{z}_1) \leqslant \tilde{G}(\hat{z}_1), \tag{22}$$

for all z_1 in $A_+ \cup A_-$. On the other hand,

$$h(z_1) + h(\hat{z}_1) \leqslant \tilde{h}(z_1) + \tilde{h}(\hat{z}_1),$$

and hence

$$\begin{aligned} e^{-\tilde{V}_1(\hat{z}_1)}\,\tilde{h}(\hat{z}_1)-e^{-V_1(z_1)}\,h(z_1) &= e^{-\tilde{V}_1(\hat{z}_1)}[\tilde{h}(\hat{z}_1)-h(z_1)] \\ &\geqslant e^{-V_1(\hat{z}_1)}[\tilde{h}(\hat{z}_1)-h(z_1)] \\ &\geqslant e^{-V_1(\hat{z}_1)}[h(\hat{z}_1)-\tilde{h}(z_1)] \\ &= e^{-V_1(\hat{z}_1)}h(\hat{z}_1)-e^{-\tilde{V}_1(z_1)}\tilde{h}(z_1). \end{aligned}$$

Therefore

$$G(z_1)+G(\hat{z}_1)\leqslant \tilde{G}(z_1)+\tilde{G}(\hat{z}_1), \tag{23}$$

for every $z_1 \in A_+ \cup A_-$. Following the argument of Lemma 2 one easily sees that, for $z_0 \in A_+$, Theorem 2 follows from (22), (23) and Lemma 1. We obtain Theorem 2 for $z_0 \in A_-$ by means of the change of variables $w_i=(-x_i,\, y_i)$ where $z_i=(x_i,\, y_i)$, $1\leqslant i\leqslant m$. ∎

3. PROOF OF THEOREM 1: INEQUALITIES FOR INTEGRALS OF HEAT KERNELS

In this section we will use our multiple integral inequalities from the previous section to prove Theorem 1. Recall that B_s is two dimensional Brownian motion and let

$$P_t^{\mathbf{R}^2}(z, w)=\frac{1}{2t\pi}\,e^{-|z-w|^2/2t},$$

$z,\ w \in \mathbf{R}^2$, be its transition density function. In one dimension we simply write

$$P_t(x, y)=\frac{1}{\sqrt{2t\pi}}\,e^{-(x-y)^2/2t},$$

for $x,\ y \in \mathbf{R}$.

By the continuity of the Brownian paths

$$\begin{aligned} &E_{(\tilde{w}_1, v_1)}\left\{\exp\left(-\int_0^t V(B_s)\,ds\right),\ \tau_D>t\right\} \\ &\quad = E_{(\tilde{w}_1, v_1)}\left\{\exp\left(-\int_0^t V(B_s)\,ds\right);\ B_s\in D,\ \forall 0\leqslant s\leqslant t\right\} \\ &\quad = \lim_{m\to\infty} E_{(\tilde{w}_1, v_1)}\left\{\exp\left(-\frac{t}{m}\sum_{i=1}^m V(B_{it/m})\right);\ B_{it/m}\in D,\ i=1, ..., m\right\}. \end{aligned}$$

By the Markov property we see that

$$E_{(\tilde{w}_1, v_1)}\left\{\exp\left(-\frac{t}{m}\sum_{i=1}^{m} V(B_{it/m})\right); B_{it/m} \in D, i=1, ..., m\right\}$$

$$= \int_{D^m} e^{-(t/m)\sum_{i=1}^{m} V(z_i)} \prod_{i=1}^{m} P_{t/m}^{\mathbf{R}^2}(z_i - z_{i-1})\, dz_1 \cdots dz_m,$$

where $z_0 = (\tilde{w}_1, v_1)$.

Let us now write the domain as $D = \{(x, y) \in \mathbf{R}^2 : y \in (-a, a), -f(y) < x < f(y)\}$ for some function f so that $b = \sup_{(x, y) \in D} |x|$, where a and b are as in the introduction. By Fubini's theorem,

$$E_{(\tilde{w}_1, v_1)}\left\{\exp\left(-\int_0^t V(B_s)\, ds\right), \tau_D > t\right\}$$

is equal to

$$\lim_{m \to \infty} \int_{(-a, a)^m} \Phi(\tilde{w}_1, y_1, ..., y_m)\, d\mu_{m, v_1}(y_1, ..., y_m),$$

where $\Phi(x_0, y_1, ..., y_m)$ is the function given by

$$\int_{-f(y_1)}^{f(y_1)} \cdots \int_{-f(y_m)}^{f(y_m)} e^{-(t/m)\sum_{i=1}^{m} V(x_i, y_i)} \prod_{i=1}^{m} P_{t/m}(x_i - x_{i-1})\, dx_1 \cdots dx_m,$$

and $\mu_{m, y_0}(y_1, ..., y_m)$ is the probability measure with density

$$\prod_{i=1}^{m} P_{t/m}(y_i - y_{i-1}).$$

Define the function $\Phi_+(x_0, y_1, ..., y_m)$ by

$$\int_0^{f(y_1)} \cdots \int_0^{f(y_m)} e^{-(t/m)\sum_{i=1}^{m} V(x_i, y_i)} \prod_{i=1}^{m} P_{t/m}(x_i - x_{i-1})\, dx_1 \cdots dx_m,$$

and set

$$\Gamma(x_0, b, m) = \int_{-b}^{b} \cdots \int_{-b}^{b} \prod_{i=1}^{m} P_{t/m}(x_i - x_{i-1})\, dx_1 \cdots dx_m,$$

and

$$\Gamma_+(x_0, b, m) = \int_0^b \cdots \int_0^b \prod_{i=1}^m P_{t/m}(x_i - x_{i-1})\, dx_1 \cdots dx_m.$$

With this notation we see that

$$E_{(w_1, v_1)} \left\{ \exp\left(-\int_0^t V(B_s)\, ds \right), \tau_{D^+} > t \right\}$$

$$= \lim_{m\to\infty} \int_{(-a,a)^m} \Phi_+(w_1, y_1, ..., y_m)\, d\mu_{m, v_1}(y_1, ..., y_m),$$

and similarly,

$$P_{\tilde{w}_2}\{\tau_{(-b,b)} > t\} = \lim_{m\to\infty} \Gamma(\tilde{w}_2, b, m),$$

and

$$P_{w_2}\{\tau_{(0,b)} > t\} = \lim_{m\to\infty} \Gamma_+(w_2, b, m).$$

Therefore it is enough to prove that

$$\Gamma(\tilde{w}_2, b, m) \int_{(-a,a)^m} \Phi_+(w_1, y_1, ..., y_m)\, d\mu_{m, v_1}(y_1, ..., y_m)$$

$$\leqslant \Gamma_+(w_2, b, m) \int_{(-a,a)^m} \Phi(\tilde{w}_1, y_1, ..., y_m)\, d\mu_{m, v_1}(y_1, ..., y_m).$$

We shall prove that

$$\Gamma(\tilde{w}_2, b, m)\, \Phi_+(w_1, y_1, ..., y_m) \leqslant \Gamma_+(w_2, b, m)\, \Phi(\tilde{w}_1, y_1, ..., y_m), \quad (24)$$

for all $y_i \in (-a, a)$ and for all $i = 1, 2, ..., m$. Let $\alpha_i = f(y_i)$ and observe that $\alpha_i < b$ by definition. Set

$$V_i(x, y) = \frac{t}{m} V(y, y_i),$$

and

$$\tilde{V}_i(x, y) = \frac{t}{m} V(x, y_i).$$

Let $z_0 = (\tilde{w}_2, w_1)$, $\tilde{z}_0 = (\tilde{w}_1, w_2)$, then proving (24) is equivalent to proving that

$$\int_{C(\alpha_1)} \cdots \int_{C(\alpha_m)} e^{-\sum_{i=1}^m V_i(z_i)} \prod_{i=1}^m P_{t/m}^{\mathbf{R}^2}(z_i - z_{i-1})\, dz_1 \cdots dz_m$$
$$\leqslant \int_{\tilde{C}(\alpha_1)} \cdots \int_{\tilde{C}(\alpha_m)} e^{-\sum_{i=1}^m \tilde{V}_i(\tilde{z}_i)} \prod_{i=1}^m P_{t/m}^{\mathbf{R}^2}(\tilde{z}_i - \tilde{z}_{i-1})\, d\tilde{z}_1 \cdots d\tilde{z}_m.$$

Clearly $P_{t/m}^{\mathbf{R}^2}(z)$ is a radially symmetric nonincreasing function, and by definition $V_i(z) = \tilde{V}_i(\hat{z})$ and $V_i(\hat{z}) = \tilde{V}_i(z)$ for all $z \in A_+ \cup A_-$. Since $V(\cdot, y)$ is nondecreasing in $\mathbf{R}^+$ we know that $V_i(\hat{z}) \geqslant \tilde{V}_i(\hat{z})$ for all $z \in A_+ \cup A_-$. The desired inequality then follows from Theorem 2 applied with $\beta = b$ and $\alpha_i = f(y_i)$. This completes the proof of Theorem 1.

The following is a direct corollary to the above proof.

Corollary 3. *Let B_t be two dimensional Brownian motion and let $0 < t_1 < t_2 < \cdots < t_m$. Suppose $0 < \alpha_i < \beta$ for all $i = 1, ..., m$. Then*

$$P_{(x,x)}\{B_{t_1} \in C(\alpha_1), B_{t_2} \in C(\alpha_2), ..., B_{t_m} \in C(\alpha_m)\}$$
$$\leqslant P_{(x,x)}\{B_{t_1} \in \tilde{C}(\alpha_1), B_{t_2} \in \tilde{C}(\alpha_2), ..., B_{t_m} \in \tilde{C}(\alpha_m)\}.$$

In particular, for all $0 < x < \alpha < \beta$ and all $t > 0$ we have

$$P_{(x,x)}\{\tau_{C(\alpha)} > t\} \leqslant P_{(x,x)}\{\tau_{\tilde{C}(\alpha)} > t\}, \tag{25}$$

where $\tau_{C(\alpha)}$ and $\tau_{\tilde{C}(\alpha)}$ denote the exit times of B_t from the regions $C(\alpha)$ and $\tilde{C}(\alpha)$, respectively. By independence, (25) is equivalent to

$$\frac{P_x\{\tau_{(0,\alpha)} > t\}}{P_x\{\tau_{(-\alpha,\alpha)} > t\}} \leqslant \frac{P_x\{\tau_{(0,\beta)} > t\}}{P_x\{\tau_{(-\beta,\beta)} > t\}}. \tag{26}$$

Since Theorem 2 holds for any sequence of radially symmetric decreasing functions, this corollary not only holds for Brownian motion but also for any other Markov right continuous processes whose transition functions have these properties. In particular we have

Corollary 4. *Let Z_t be a two dimensional symmetric stable processes and let $0 < t_1 < t_2 < \cdots < t_m$. Suppose $0 < \alpha_i < \beta$ for all $i = 1, ..., m$. Then*

$$P_{(x,x)}\{Z_{t_1} \in C(\alpha_1), Z_{t_2} \in C(\alpha_2), ..., Z_{t_m} \in C(\alpha_m)\}$$
$$\leqslant P_{(x,x)}\{Z_{t_1} \in \tilde{C}(\alpha_1), Z_{t_2} \in \tilde{C}(\alpha_2), ..., Z_{t_m} \in \tilde{C}(\alpha_m)\}.$$

In particular, for all $0 < x < \alpha < \beta$ *and all* $t > 0$ *we have*

$$P_{(x,x)}\{\eta_{C(\alpha)} > t\} \leqslant P_{(x,x)}\{\eta_{\tilde{C}(\alpha)} > t\},$$

where $\eta_{C(\alpha)}$ *and* $\eta_{\tilde{C}(\alpha)}$ *denote the exit times of* Z_t *from the regions* $C(\alpha)$ *and* $\tilde{C}(\alpha)$, *respectively.*

This time, however, we cannot state the analogue of (26) since the coordinates are not independent.

4. PROOFS OF COROLLARIES 1 AND 2 AND CONSEQUENCES FOR NODAL LINES

It follows from the eigenfunction expansion of the heat kernel that for any bounded domain D and any $z \in D$,

$$\lim_{t \to \infty} \frac{1}{t} \log\left(E_z\left\{\exp\left(-\int_0^t V(B_s)\,ds\right), \tau_D > t\right\}\right) = -\beta_{1,D}^V, \tag{27}$$

where $\beta_{1,D}^V$ is the first Dirichlet eigenvalue for the operator $-\frac{1}{2}\Delta + V$ in D. This in fact is just the famous representation of the eigenvalue given by M. Kac. We refer the reader to B. Simon [19]. Set $I = (-b, b)$, $I^+ = (0, b)$ and recall that the first Dirichlet eigenvalues of the Laplacian for these intervals are denoted by $\lambda_{1,I}$ and λ_{1,I^+}, respectively, and that the second eigenvalue of I is denoted by $\lambda_{2,I}$. From Theorem 1 and (27) we have

COROLLARY 5. *Suppose* D, V, *and* l *are as in Corollary* 1. *Then*

$$\lambda_{1,D^+}^V - \lambda_{1,D}^V \geqslant \lambda_{1,I^+} - \lambda_{1,I} = \lambda_{2,I} - \lambda_{1,I} = \frac{3\pi^2}{l^2}.$$

We shall now see that Corollary 1 follows from Corollary 5. By the Courant nodal domain theorem, a nodal curve, which is the set of all points in the domain where an eigenfunction with eigenvalue $\lambda_{2,D}$ vanishes, divides the domain into two subdomains. These subdomains are called nodal domains. The second Dirichlet eigenvalue of Δ in D is the first Dirichlet eigenvalue of Δ in either of these two nodal domains. The study of the geometry of nodal curves has been of interest for many years but surprisingly very little is known for general domains. However, in the case of planar domains which are symmetric with respect to both coordinate axes and convex in both axes, L. Payne [17] proved that there are no closed nodal curves and that there exists an eigenfunction whose nodal curve (nodal line in this case) is the intersection of the domain with one of the coordinate axes. Assume now that $V = 0$ and that D satisfies the hypothesis of Corollary 1. In this case $2b$ is just the length of the intersection of the x-axis with the domain and $2a$ is just the length of the intersection of the y-axis with the domain. Recall that we assumed that the length of the major axes is $2b$. That is, $b > a$. If the nodal line is on the y-axis then D^+ is a nodal domain, $\lambda_{1, D^+} = \lambda_{2, D}$, and Corollary 1 follows from Corollary 5. If the nodal line is on the x-axes, we rotate the domain and represent it by $D = \{(x, y) \in \mathbf{R}^2 : y \in (-b, b),\ -\tilde{f}(y) < x < \tilde{f}(y)\}$. Interchanging the roles of a and b in Theorem 1 we have that

$$\lambda_{1, D^+} - \lambda_{1, D} \geqslant \frac{3\pi^2}{4a^2},$$

and as above this implies that

$$\lambda_{2, D} - \lambda_{1, D} \geqslant \frac{3\pi^2}{4a^2}.$$

Notice, however, that this time the result is even better since we assumed $a < b$ and hence $4a^2 < 4b^2 = l^2$. With this, Corollary 1 is proved.

We now prove Corollary 2. Letting $a \to 0$ we obtain from Corollary 5 that

$$\lambda^V_{1, I^+} - \lambda^V_{1, I} \geqslant \lambda_{1, I^+} - \lambda_{1, I} = \lambda_{2, I} - \lambda_{1, I} = \frac{3\pi^2}{l^2}.$$

It remains to verify that under the assumptions of Corollary 2,

$$\lambda^V_{1, I^+} = \lambda^V_{2, I}.$$

Let φ be an eigenfunction with eigenvalue $\lambda^V_{2, I}$ and set $w(x) = \varphi(x) + \varphi(-x)$. Since V is an even function we have that w is an even eigenfunction with eigenvalue $\lambda^V_{2, I}$. Let $N(w) = \{x \in I : w(x) = 0\}$. It is clear that if $x \in N(w)$ then $-x \in N(w)$. Then, by the Courant nodal domain theorem, $N(w) = \{0\}$ unless $w = 0$. But $N(w) = \{0\}$ implies that w does not change sign on I, which contradicts the fact that w is orthogonal to the first eigenfunction. Thus φ is odd and I^+ is a nodal domain. Hence, $\lambda^V_{1, I^+} = \lambda^V_{2, I}$ and Corollary 2 follows.

We end this section with some consequences of the above arguments for the location of the nodal line and the multiplicity of the second eigenvalue for convex domains which are symmetric and "long enough." More precisely we have

COROLLARY 6. *Let $D \subset \mathbf{R}^2$ be a bounded convex domain which is symmetric with respect to both coordinate axes. Let d_D be its diameter and r_D*

be its inner radius. There is an absolute constant C such that whenever $d_D/r_D > C$, then the multiplicity of the second eigenvalue is one and the nodal line lies on the y-axis.

Proof. Smits [21] has proved that for any convex domain in the plane there is an absolute constant C_1 such that

$$\lambda_{2,D} - \lambda_{1,D} \leqslant \frac{C_1}{d_D^{2/3} r_D^{4/3}}. \tag{28}$$

Suppose now that in addition D is symmetric with respect to both coordinate axes. By Payne's result [17] one of the two axis of symmetry is a nodal line for a second eigenfunction. Let a be as in the proof of Corollary 1. It follows from the proof of Corollary 1 that if there is a nodal line on the x-axis then

$$\lambda_{2,D} - \lambda_{1,D} \geqslant \frac{3\pi^2}{4a^2}. \tag{29}$$

A simple geometric argument shows that $a/2\sqrt{2} \leqslant r_D \leqslant a$. This and the inequalities (28) and (29) give that

$$\frac{d_D}{r_D} \leqslant \left(\frac{8\sqrt{2}\, C_1}{3\pi^2}\right)^{2/3}.$$

Hence if

$$\frac{d_D}{r_D} > \left(\frac{8\sqrt{2}\, C_1}{3\pi^2}\right)^{2/3} \tag{30}$$

there is no nodal line on the x-axis.

We shall now prove that under (30) the multiplicity of $\lambda_{2,D}$ is one. Towards this end, let φ be any eigenfunction with eigenvalue $\lambda_{2,D}$. Set

$$u(x, y) = \varphi(x, y) + \varphi(-x, y).$$

Then u is a second eigenfunction with eigenvalue $\lambda_{2,D}$ and it satisfies the property that for all $(x, y) \in D$, $u(x, y) = u(-x, y)$. Suppose u is not identically zero. Consider the new eigenfunction with eigenvalue $\lambda_{2,D}$

$$v(x, y) = u(x, y) + u(x, -y).$$

This eigenfunction satisfies

$$v(x, y) = v(-x, y) = v(x, -y)$$

for all $(x, y) \in D$. It follows from this that unless v is identically zero, it has a closed nodal curve which is a contradiction to L. Payne's result [17]. Therefore the function v is identically zero. Hence $u(x, -y) = -u(x, y)$ and thus its nodal line is on the x-axis. However, by what we have already proved if the inequality (30) is satisfied there is no second eigenfunction with its nodal line along the x-axis. Thus the function u must also be identically zero. That is, under (30) we must have that

$$\varphi(x, y) = -\varphi(-x, y)$$

for any eigenfunction corresponding to $\lambda_{2, D}$. Thus every second eigenfunction must have its nodal line along the y-axis and hence the multiplicity of the second eigenvalue is one by [18, Lemma 1]. This completes the proof. ▮

Remark 1. It is known (see C. S. Lin [14]) that for any bounded smooth convex domain in $\mathbf{R}^2$ the multiplicity of the second eigenvalue is at most two. Also, the above Corollary 6 should be compared with the corresponding result for the Neumann eigenfunctions proved in Bañuelos and Burdzy [6, Proposition 2.4].

5. A SCHRÖDINGER OPERATOR WITH A CLOSED NODAL LINE

In the previous section we showed how Corollary 1 follows from Corollary 5 by showing that for our domains, $\lambda_{1, D^+} = \lambda_{2, D}$. Corollary 5 would immediately imply a version of Corollary 1 for nonzero potentials if it were true that $\lambda^V_{1, D^+} = \lambda^V_{2, D}$. Unfortunately, this last assertion is false for our class of potentials even when the domain D is a disk. We shall now proceed to give an example.

Let $D_r = B(0, r) \subset \mathbf{R}^2$ the open disk of radius r centered at the origin 0. In [15], Lin and Ni prove that there exist functions f, $\hat{u}$ and $\hat{V}$, and a real number $r > 0$, with the following properties:

(i) $f: \mathbf{R} \to \mathbf{R}$ is a smooth convex function with $f(x) > 0$ for $x > 0$ and $f(0) = f'(0) = 0$.

(ii) $\hat{u} \in C^2(\bar{D}_r)$ is a positive solution to the problem

$$\begin{cases} \Delta u + f(u) = 0, & \text{in } D_r \\ u = 0, & \text{on } \partial D_r. \end{cases} \tag{31}$$

(iii) Let $\hat{V} = -f'(\hat{u})$, then any second eigenfunction of the problem

$$\begin{cases} -\Delta\varphi + \hat{V}\varphi = \lambda\varphi, & \text{in } D_r \\ \varphi = 0, & \text{on } \partial D_r \end{cases} \tag{32}$$

is radial.

Note that $\hat{V} = -f'(\hat{u})$ is a bounded potential in D_r, since f is smooth in $\mathbf{R}$ and $\hat{u}$ is continuous in $\bar{D}_r$. Thus the spectrum of $-\Delta + \hat{V}$ on D_r is discrete (see [10, Lemmas 1.6.5 and 1.6.8]).

Let φ_2 be any second eigenfunction of (32) and $\lambda_{2, D_r}^{\hat{V}}$ be its eigenvalue. It follows from (iii) that its nodal curve $N(\varphi_2)$ is a circle. If $N(\varphi_2) = \partial B_l$, $l < r$, then by the Courant nodal domain theorem $\lambda_{2, D_r}^{\hat{V}}$ is the first eigenvalue of the operator $-\Delta + \hat{V}$ on D_l. Hence l is uniquely determine by $\lambda_{2, D_r}^{\hat{V}}$, and any two second eigenfunctions must have the same nodal line. By [18, Lemma 1] we conclude that the multiplicity of $\lambda_{2, D_r}^{\hat{V}}$ is one.

It was proved in [11] that any solution of (31) is radially decreasing. Thus we have

$$\frac{\partial \hat{V}}{\partial r} = -f''(\hat{u}) \frac{\partial \hat{u}}{\partial r} \geqslant 0.$$

Let

$$|\hat{V}|_\infty = \sup_{y \in D_r} |\hat{V}(y)|$$

and consider

$$V = |\hat{V}|_\infty + \hat{V}.$$

Then V is bounded, nonnegative, and radially increasing, hence it satisfies the conditions of Theorem 1.

It is easy to prove that $\lambda_{k, D_r}^{V} = \lambda_{k, D_r}^{\hat{V}} + |\hat{V}|_\infty$, and that an eigenfunction of eigenvalue λ_{k, D_r}^{V} is an eigenfunction of eigenvalue $\lambda_{k, D_r}^{\hat{V}}$. Therefore the nodal line of any eigenfunction of eigenvalue $\lambda_{2, D}^{V}$ is a circle. Moreover, if φ_1 is the eigenfunction of eigenvalue λ_{1, D^+}^{V}, it is easy to prove that $\varphi(x, y) = \varphi_1(x, y)\, I(x \geqslant 0) - \varphi_1(-x, y)\, I(x \leqslant 0)$ satisfies

$$\begin{cases} -\Delta\varphi + V\varphi = \lambda_{1, D^+}^{V}\varphi, & \text{in } D_r \\ \varphi = 0, & \text{on } \partial D_r. \end{cases}$$

Thus there exists $k \geqslant 1$ such that $\lambda^V_{1, D_r^+} = \lambda^V_{k, D_r}$. Note that the nodal line of φ is the intersection of the y-axis with the domain D_r. Since the eigenfunction of eigenvalue λ^V_{1, D_r} does not vanish in D_r, and the nodal line of any eigenfunction of eigenvalue λ^V_{2, D_r} is a circle, we have that $k > 2$, and hence $\lambda^V_{1, D_r^+} > \lambda^V_{2, D_r}$. We summarize the above discussion in the following theorem.

THEOREM 3. *There exist $r > 0$ and a nonnegative bounded potential V on $D_r = B(0, r)$ which is radial and increasing, hence satisfying the hypothesis of Theorem 1, for which the second Dirichlet eigenvalue λ^V_{2, D_r} of $-\Delta + V$ in D_r is simple and the nodal line of its corresponding eigenfunction is a circle. Furthermore, $\lambda^V_{1, D_r^+} > \lambda^V_{2, D_r}$.*

We close this section by showing that under additional assumptions, in particular convexity, the nodal line is on one of the coordinate axes.

Let W be a radial potential on $D_r = B(0, r)$ such that $[\rho W(\rho)]'' \geqslant 0$ for $0 \leqslant \rho \leqslant r$. Since any radial convex potential in D_r is radially increasing, this class of operators contains all the radial convex potentials in D_r. Recall that in polar coordinates $-\Delta + W$ is given by

$$-\frac{\partial^2}{\partial \rho^2} - \frac{n-1}{\rho}\frac{\partial}{\partial \rho} + W(r) - \frac{1}{\rho^2}\Delta_{r\mathbf{S}^{n-1}},$$

where $\Delta_{r\mathbf{S}^{n-1}}$ is the Laplace–Beltrami operator in $r\mathbf{S}^{n-1}$ and $\rho = (y_1^2 + \cdots + y_n^2)^{1/2}$. It is well known that the eigenvalues of $-\Delta_{r\mathbf{S}^{n-1}}$ are $\{i(i+n-2)\}_{i=0}^{\infty}$. Let $\{\psi_i\}_{i=0}^{\infty}$ be the corresponding set of eigenfunctions, that is the sequences is such that

$$\begin{cases} -\Delta_{r\mathbf{S}^{n-1}}\psi_i = i(i+n-2)\,\psi & \text{in} \quad r\mathbf{S}^{n-1} \\ \psi_i = 0, & \text{on} \quad \partial r\mathbf{S}^{n-1}. \end{cases}$$

Then any eigenfunction of $-\Delta + W$ in D_r is of the form $f(\rho)\,\psi_i(\xi)$, where $f(r)$ is an eigenfunction of the operator

$$H_i = \frac{\partial^2}{\partial \rho^2} + \frac{n-1}{\rho}\frac{\partial}{\partial \rho} + W(r) + \frac{i(i+n-2)}{\rho^2},$$

in $\mathbf{R}^+$. Consequently, if $\lambda_k^{H_i}$ is the set of eigenvalues of the operator H_i in $\mathbf{R}^+$, then $\{\lambda_k^{H_i}\}_{i=0, k=1}^{\infty}$ are the eigenvalues of $-\Delta + W$ in D_r.

It is clear that $\lambda_k^{H_i} \leqslant \lambda_k^{H_{i+1}}$ for $i \geqslant 0$ and $k \geqslant 1$. On the other hand, Theorem 3.2 in [1] states that if $[\rho W(\rho)]'' \geqslant 0$ for all $0 \leqslant \rho \leqslant r$, then $\lambda_k^{H_{i+1}} \leqslant \lambda_{k+1}^{H_i}$ for all $i \geqslant 0$ and $k \geqslant 1$. Therefore

$$\lambda^W_{1, D_r} = \lambda_1^{H_1}, \qquad \lambda^W_{2, D_r} = \lambda_1^{H_2},$$

and any second eigenfunction of $-\Delta + W$ is of the form $f(r)\,\psi_2(\xi)$, where ψ_2 is a second eigenfunction of $-\Delta_{r\mathbf{S}^{n-1}}$ and $f(r)$ is the first eigenfunction of H_2. Since $f(r) > 0$ and the nodal line of $\psi_2(\xi)$ is contained in one of the coordinates axis; we conclude that any nodal curve of a second eigenfunction of $-\Delta + W$ is contained in one of the coordinate axis.

Therefore if V is the potential of Theorem 3, there exists $\rho \in (0, r)$ such that $[\rho V(\rho)]'' < 0$. In particular V is nonconvex.

6. POINTWISE INEQUALITIES FOR HEAT KERNELS

Let us recall that $P_V^D(t, z, w)$ and $P_V^{D^+}(t, z, w)$ are the Dirichlet heat kernels of the operator $-\frac{1}{2}\Delta + V$ in D and D^+, respectively. Let $P^C(t, z, w)$, $P^{C^+}(t, z, w)$, $P^{(-b,b)}(t, x, y)$, and $P^{(0,b)}(t, x, y)$ be the Dirichlet heat kernels of the Laplacian (i.e., zero potentials) in C, C^+, $(-b, b)$ and $(0, b)$, respectively.

THEOREM 4. *Suppose D and V are as in Theorem 1. Let (w_1, v), $(v_1, v_0) \in D^+$, $(\tilde{w}_1, v)$, $(\tilde{v}_1, v_0) \in D$, $(w_2, \tilde{v})$, $(v_2, \tilde{v}_0) \in C^+$, and $(\tilde{w}_2, \tilde{v})$, $(\tilde{v}_2, \tilde{v}_0) \in C$. Suppose $z = (\tilde{w}_2, w_1)$ and $z_0 = (\tilde{v}_2, v_1)$ are such that $\hat{z}_0 = (\tilde{v}_1, v_2)$ and*

- *if $z \in A_+ \cup A_-$, then $\hat{z} = (\tilde{w}_1, w_2)$ and*
- *if $z \notin A_+ \cup A_-$, then $\tilde{w}_2 = \tilde{w}_1$ and $w_2 = w_1$.*

Then for all $t > 0$,

$$\frac{P_V^{D^+}(t, (w_1, v), (v_1, v_0))}{P_V^D(t, (\tilde{w}_1, v), (\tilde{v}_1, v_0))} \leqslant \frac{P^{C^+}(t, (w_2, \tilde{v}), (v_2, \tilde{v}_0))}{P^C(t, (\tilde{w}_2, \tilde{v}), (\tilde{v}_2, \tilde{v}_0))} = \frac{P^{(0,b)}(t, w_2, v_2)}{P^{(-b,b)}(t, \tilde{w}_2, \tilde{v}_2)}. \tag{33}$$

In particular, for all $z, w \in D^+$ and all $t > 0$,

$$\frac{P_V^{D^+}(t, z, w)}{P_V^D(t, z, w)} \leqslant \frac{P^{C^+}(t, z, w)}{P^C(t, z, w)}, \tag{34}$$

and

$$\frac{P_V^{D^+}(t, z, w)}{P_V^D(t, z, w^*)} \leqslant \frac{P^{C^+}(t, z, w)}{P^C(t, z, w^*)}, \tag{35}$$

where w^* is the reflection of w with respect to the y-axis.

An immediate corollary of (34) and (35) is

COROLLARY 7. *Suppose D and V are as in Theorem 1. For all $(z, w) \in D^+$ and all $t > 0$,*

$$\frac{P_V^{D^+}(t, z, w)}{P_V^D(t, z, w) + P_V^D(t, z, w^*)} \leqslant \frac{P^{C^+}(t, z, w)}{P^C(t, z, w) + P^C(t, , z, w^*)}.$$

This corollary gives Davis' inequality (5) upon taking $V = 0$.

The proof of Theorem 3 is very similar to the proof of Theorem 1. By the Feynman–Kac formula (see Simon [19]) the solutions of the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2} \Delta u - Vu, & \text{in } D \\ u(t, \cdot) = 0 & \text{on } \partial D \\ u(0, z) = f(z) & \text{in } D \end{cases}$$

have the representation (for suitable f's)

$$\begin{aligned} u(t, z) &= E_z \left\{ f(B_t) \exp\left(- \int_0^t V(B_s)\, ds \right), \tau_D > t \right\} \\ &= E_z \left[f(B_t)\, E_z \left\{ \exp\left(- \int_0^t V(B_s)\, ds \right), \tau_D > t \mid B_t \right\} \right] \\ &= \int_D P_t^{\mathbf{R}^2}(z, w)\, f(w)\, E_z \left\{ \exp\left(- \int_0^t V(B_s)\, ds \right), \tau_D > t \mid B_t = w \right\} dw. \end{aligned}$$

Thus

$$P_V^D(t, z, w) = P_t^{\mathbf{R}^2}(z, w)\, E_z \left\{ \exp\left(- \int_0^t V(B_s)\, ds \right), \tau_D > t \mid B_t = w \right\},$$

with a similar representation for $P_V^{D^+}(t, z, w)$, $P^C(t, z, w)$, $P^{C^+}(t, z, w)$, $P^{(-b, b)}(t, x, y)$ and $P^{(0, b)}(t, x, y)$. As in the proof of Theorem 1 we have

$$\begin{aligned} &E_z \left\{ \exp\left(- \int_0^t V(B_s)\, ds \right), \tau_D > t \mid B_t = w \right\} \\ &\quad = E_z \left\{ \exp\left(- \int_0^t V(B_s)\, ds \right); B_s \in D, \forall 0 \leqslant s \leqslant t \mid B_t = w \right\} \\ &\quad = \lim_{m \to \infty} E_z \left\{ \exp\left(- \frac{t}{m} \sum_{i=1}^{m-1} V(B_{it/m}) \right); B_{it/m} \in D, i = 1, ..., m-1 \mid B_t = w \right\}. \end{aligned}$$

Let $0=t_0<t_1<\cdots<t_n<t$, then the conditional finite-dimensional distribution

$$P_{z_0}\{B_{t_1}\in dz_1,\ldots,B_{t_n}\in dz_n \mid B_t=w\},$$

is given by

$$\frac{P^{\mathbf{R}^2}_{t-t_n}(z_n,w)}{P^{\mathbf{R}^2}_t(z_0,w)}\prod_{i=1}^{n}P^{\mathbf{R}^2}_{t_i-t_{i-1}}(z_i,z_{i-1}),$$

see [12, p. 359]. Hence for $m\geqslant 2$

$$P^{\mathbf{R}^2}_t(z,w)\,E_z\left\{\exp\left(-\frac{t}{m}\sum_{i=1}^{m-1}V(B_{it/m})\right);B_{it/m}\in D,\,i=1,\ldots,m-1\mid B_t=w\right\}$$

$$=\int_{D^{m-1}}e^{-(t/m)\sum_{i=1}^{m-1}V(z_i)}\prod_{i=1}^{m}P^{\mathbf{R}^2}_{t/m}(z_i-z_{i-1})\,dz_1\cdots dz_{m-1},$$

where $z_0=z$ and $z_m=w$. Therefore Fubini's theorem gives that

$$P^D_V(t,(\tilde{w}_1,v),(\tilde{v}_1,v_0))$$

$$=\lim_{m\to\infty}\int_{(-a,a)^{m-1}}\Phi(\tilde{w}_1,\tilde{v}_1,y_1,\ldots,y_{m-1})\,d\mu_{m,v,v_0}(y_1,\ldots,y_{m-1}),$$

where $\Phi(x_0,x_m,y_1,\ldots,y_{m-1})$ is the function

$$\int_{-f(y_1)}^{f(y_1)}\cdots\int_{-f(y_{m-1})}^{f(y_{m-1})}e^{-(t/m)\sum_{i=1}^{m-1}V(x_i,y_i)}\prod_{i=1}^{m}P_{t/m}(x_i-x_{i-1})\,dx_1\cdots dx_{m-1},$$

and $\mu_{m,y_0,y_m}(y_1,\ldots,y_{m-1})$ is the measure

$$\prod_{i=1}^{m}P_{t/m}(y_i-y_{i-1})\,dy_1\cdots dy_{m-1}.$$

Define the function $\Phi_+(x_0,x_m,y_1,\ldots,y_{m-1})$ by

$$\int_{0}^{f(y_1)}\cdots\int_{0}^{f(y_{m-1})}e^{-(t/m)\sum_{i=1}^{m-1}V(x_i,y_i)}\prod_{i=1}^{m}P_{t/m}(x_i-x_{i-1})\,dx_1\cdots dx_{m-1},$$

and set

$$\Gamma(x_0,x_m,b,m)=\int_{-b}^{b}\cdots\int_{-b}^{b}\prod_{i=1}^{m}P_{t/m}(x_i-x_{i-1})\,dx_1\cdots dx_{m-1},$$

and

$$\Gamma_+(x_0, x_m, b, m) = \int_0^b \cdots \int_0^b \prod_{i=1}^m P_{t/m}(x_i - x_{i-1})\, dx_1 \cdots dx_{m-1}.$$

With this notation we see that

$$\begin{aligned} P_V^{D^+}(t, (w_1, v), (v_1, v_0)) &= \lim_{m\to\infty} \int_{(-a,a)^{m-1}} \Phi_+(w_1, v_1, y_1, ..., y_{m-1}) \\ &\qquad \times d\mu_{m, v, v_0}(y_1, ..., y_{m-1}), \\ P^{(-b,b)}(t, \tilde{w}_2, \tilde{v}_2) &= \lim_{m\to\infty} \Gamma(\tilde{w}_2, \tilde{v}_2, b, m), \end{aligned}$$

and

$$P^{(0,b)}(t, w_2, v_2) = \lim_{m\to\infty} \Gamma_+(w_2, v_2, b, m).$$

Therefore it is enough to prove that

$$\begin{aligned} &\Gamma(\tilde{w}_2, \tilde{v}_2, b, m) \int_{(-a,a)^{m-1}} \Phi_+(w_1, v_1, y_1, ..., y_{m-1})\, d\mu_{m, v, v_0}(y_1, ..., y_{m-1}) \\ &\quad \leqslant \Gamma_+(w_2, v_2, b, m) \int_{(-a,a)^{m-1}} \Phi(\tilde{w}_1, \tilde{v}_1, y_1, ..., y_{m-1}) \\ &\qquad \times d\mu_{m, v, v_0}(y_1, ..., y_{m-1}). \end{aligned}$$

We shall prove that

$$\begin{aligned} &\Gamma(\tilde{w}_2, \tilde{v}_2, b, m)\, \Phi_+(w_1, v_1, y_1, ..., y_{m-1}) \\ &\quad \leqslant \Gamma_+(w_2, v_2, b, m)\, \Phi(\tilde{w}_1, \tilde{v}_1, y_1, ..., y_{m-1}), \end{aligned} \tag{36}$$

for all $y_i \in (-a, a)$ and for all $i = 1, 2, ..., m-1$. As before, let $\alpha_i = f(y_i)$ and set

$$V_i(x, y) = \frac{t}{m} V(y, y_i),$$

and

$$\tilde{V}_i(x, y) = \frac{t}{m} V(x, y_i).$$

Let $z_0=(\tilde{w}_2, w_1)$, $z_m=(\tilde{v}_2, v_1)$, $\tilde{z}_0=(\tilde{w}_1, w_2)$, $\tilde{z}_m=(\tilde{v}_1, v_2)$, thus proving (36) is equivalent to proving that

$$\int_{C(\alpha_1)} \cdots \int_{C(\alpha_{m-1})} e^{-\Sigma_{i=1}^{m-1} V_i(z_i)} \prod_{i=1}^{m} P_{t/m}^{\mathbf{R}^2}(z_i - z_{i-1})\, dz_1 \cdots dz_{m-1}$$

$$\leqslant \int_{\tilde{C}(\alpha_1)} \cdots \int_{\tilde{C}(\alpha_{m-1})} e^{-\Sigma_{i=1}^{m-1} \tilde{V}_i(\tilde{z}_i)} \prod_{i=1}^{m} P_{t/m}^{\mathbf{R}^2}(\tilde{z}_i - \tilde{z}_{i-1})\, d\tilde{z}_1 \cdots d\tilde{z}_{m-1},$$

where $C(\alpha)$ and $\tilde{C}(\alpha)$ are as in Theorem 2. As in the proof of Theorem 1 this last assertion follows from.

THEOREM 5. *Let $\alpha_1, ..., \alpha_m$ be such that $0<\alpha_i<\beta$ for $1\leqslant i\leqslant m$ and let $p_i(z)$ be radially symmetric nonincreasing functions in $\mathbf{R}^2$ for $1\leqslant i\leqslant m+1$. Let $V_i(x, y)$, $\tilde{V}_i(x, y)$, $1\leqslant i\leqslant m$, be positive real valued functions with the property that*

- $V_i(x, y)=V_i(-x, y)$ *and* $\tilde{V}_i(x, y)=\tilde{V}_i(-x, y)$, *for all* $(x, y)\in\mathbf{R}^2$,
- $V_i(z)=\tilde{V}_i(\hat{z})$ *and* $\tilde{V}_i(z)=V_i(\hat{z})$ *for all* $z\in A_+\cup A_-$,
- $\tilde{V}_i(\hat{z})\leqslant V_i(\hat{z})$ *for all* $z\in A_+\cup A_-$.

Define the functions

$$\Upsilon_m(z_0, z_{m+1})=\int_{C(\alpha_1)} \cdots \int_{C(\alpha_m)} e^{-\Sigma_{i=1}^{m} V_i(z_i)} \prod_{i=1}^{m+1} p_i(z_i - z_{i-1})\, dz_1 \cdots dz_m$$

$$\tilde{\Upsilon}_m(z_0, z_{m+1})=\int_{\tilde{C}(\alpha_1)} \cdots \int_{\tilde{C}(\alpha_m)} e^{-\Sigma_{i=1}^{m} \tilde{V}_i(z_i)} \prod_{i=1}^{m+1} p_i(z_i - z_{i-1})\, dz_1 \cdots dz_m.$$

Then for $z_0, w_0\in A_+\cup A_-$,

$$\Upsilon_m(z_0, w_0)\leqslant\tilde{\Upsilon}_m(\hat{z}_0, \hat{w}_0),$$

$$\Upsilon_m(\hat{z}, w_0)\leqslant\tilde{\Upsilon}_m(\hat{z}_0, \hat{w}_0),$$

and

$$\Upsilon_m(z_0, w_0)+\Upsilon_m(\hat{z}_0, w_0)\leqslant\tilde{\Upsilon}_m(z_0, \hat{w}_0)+\Upsilon_m(\hat{z}_0, \hat{w}_0).$$

Following the argument of Theorem 2, it is enough to prove the following lemma which as before is just the case $m=1$.

LEMMA 3. *Fix $z, w \in A_+ \cup A_-$. Then*

$$\int_{B(z,r)\cap C(\alpha_1)\cap B(w,l)} e^{-V_1(z_1)}\,dz_1 \leqslant \int_{B(\hat{z},r)\cap \tilde{C}(\alpha_1)\cap B(\hat{w},l)} e^{-\tilde{V}_1(z_1)}\,dz_1,$$

$$\int_{B(\hat{z},r)\cap C(\alpha_1)\cap B(w,l)} e^{-V_1(z_1)}\,dz_1 \leqslant \int_{B(\hat{z},r)\cap \tilde{C}(\alpha_1)\cap B(\hat{w},l)} e^{-\tilde{V}_1(z_1)}\,dz_1,$$

and

$$\int_{B(z,r)\cap C(\alpha_1)\cap B(w,l)} e^{-V_1(z_1)}\,dz_1 + \int_{B(\hat{z},r)\cap C(\alpha_1)\cap B(w,l)} e^{-V_1(z_1)}\,dz_1$$
$$\leqslant \int_{B(z,r)\cap \tilde{C}(\alpha_1)\cap B(\hat{w},l)} e^{-\tilde{V}_1(z_1)}\,dz_1 + \int_{B(\hat{z},r)\cap \tilde{C}(\alpha_1)\cap B(\hat{w},l)} e^{-\tilde{V}_1(z_1)}\,dz_1,$$

for every $l, r > 0$.

Proof. Consider the functions

$$G(z_1) = e^{-V_1(z_1)} p_2(z_1 - w),$$

and

$$\tilde{G}(z_1) = e^{-\tilde{V}_1(z_1)} p_2(z_1 - \hat{w}),$$

where $p_2(z) = I_{B(0,l)}(z)$.

Let $z_1 \in A_+ \cup A_-$. By Lemma 1

$$p_2(z_1 - w) \leqslant p_2(\hat{z}_1 - \hat{w}), \qquad \text{and} \qquad p_2(\hat{z}_1 - w) \leqslant p_2(\hat{z}_1 - \hat{w}).$$

Since $V_1(z_1) = \tilde{V}_1(\hat{z}_1)$ and $\tilde{V}_1(\hat{z}_1) \leqslant V_1(\hat{z}_1)$, we have that

$$G(z_1) \leqslant \tilde{G}(\hat{z}_1) \qquad \text{and} \qquad G(\hat{z}_1) \leqslant \tilde{G}(\hat{z}_1). \tag{37}$$

We claim that

$$e^{-V_1(z_1)}\, p_2(z_1 - w) + e^{-V_1(\hat{z}_1)} p_2(\hat{z}_1 - w)$$
$$\leqslant e^{-\tilde{V}_1(z_1)} p_2(z_1 - \hat{w}) + e^{-\tilde{V}_1(\hat{z}_1)} p_2(\hat{z}_1 - \hat{w}). \tag{38}$$

Recall that $\tilde{V}_1(z_1) = V_1(\hat{z}_1)$ and hence

$$e^{-V_1(z_1)} + e^{-V_1(\hat{z}_1)} = e^{-\tilde{V}_1(z_1)} + e^{-\tilde{V}_1(\hat{z}_1)}. \tag{39}$$

If $z_1, w \in A_+$ or $z_1, w \in A_-$, Lemma 1 implies that

$$p_2(z_1 - w) = p_2(\hat{z}_1 - \hat{w}) \geqslant p_2(\hat{z}_1 - w) = p_2(z_1 - \hat{w}).$$

Thus it suffices to consider two cases:

- If $p_2(\hat{z}_1 - w) = 1$, then $p_2(z_1 - w) = 1$ and (38) follows from (39).
- If $p_2(z_1 - w) = 1$ and $p_1(\hat{z}_1 - w) = 0$, then (38) follows from (37).

Suppose now that $z_1 \in A_-$ and $w \in A_+$, or $z_1 \in A_+$ and $w \in A_-$. Then Lemma 1 implies that

$$p_2(z_1 - w) \leqslant p_2(\hat{z}_1 - w) \leqslant p_2(\hat{z}_1 - \hat{w}),$$

and

$$p_2(z_1 - w) \leqslant p_2(z_1 - \hat{w}) \leqslant p_2(\hat{z}_1 - \hat{w}),$$

and again two cases arise:

- If $p_2(z_1 - w) = 1$, then (38) follows from (39).
- If $p_2(z_1 - w) = 0$ and $p_1(\hat{z}_1 - w) = 1$, then (38) follows from (37).

Thus

$$G(z_1) + G(\hat{z}_1) \leqslant \widetilde{G}(z_1) + \widetilde{G}(\hat{z}_1),$$

and from here the proof proceeds exactly as the proof of Lemma 2. ∎

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