Brownian couplings – my favorite open problems. II. Synchronous couplings

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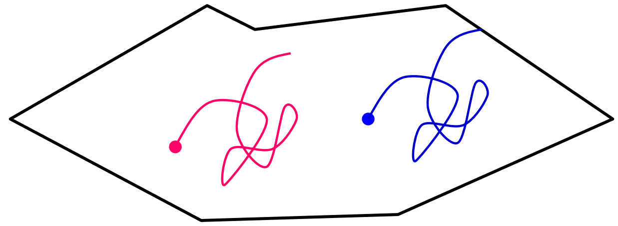
Synchronous couplings of reflected Brownian motions



Synchronous couplings via the Skorohod equation

$$X_t = x + B_t + \int_0^t N(X_s)\,dL_t^X$$

$$Y_t = y + B_t + \int_0^t N(Y_s)\,dL_t^Y$$

Synchronous couplings in smooth domains

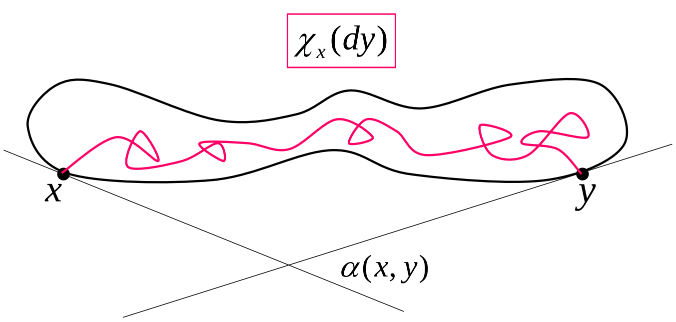
B, Chen and Jones (2006)

If $\Lambda(D) > 0$ then, a.s.,

$$\lim_{t\to\infty} \frac{\log dist(X_t, Y_t)}{t} = -\frac{\Lambda(D)}{2|D|}$$

$$dist(X_t, Y_t) \approx \exp(-\Lambda(D)t/(2|D|))$$

Cranston & Le Jan (1989, 1990)



Lyapunov exponent

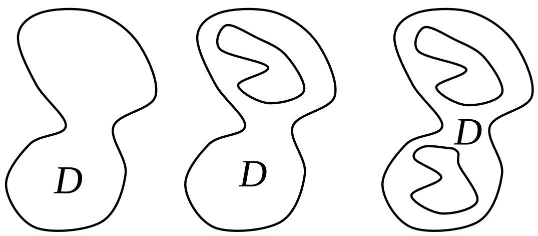
Angle between tangent lines - $\alpha(x,y)$
Curvature - $\upsilon(x)$
Harmonic measure - $\chi_x(dy)$

$$\Lambda(D) = \int_{\partial D} \upsilon(x)\,dx + \iint_{\partial D \times \partial D} |\log\cos\alpha(x,y)|\,\chi_x(dy)\,dx$$

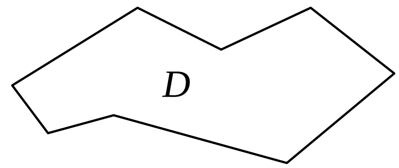
By Gauss-Bonnet Theorem,

$(1/2\pi)\int_{\partial D} \upsilon(x)\,dx = 1$ - number of holes

Corollary : If a smooth domain has at most one hole then synchronous couplings converge.

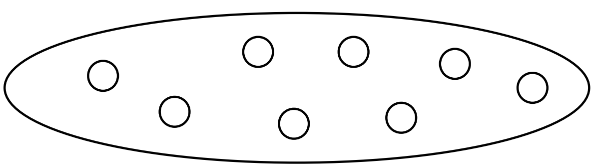


Scale invariance: $\Lambda(D) = \Lambda(cD)$
Domains with corners: $\Lambda(D) = \infty$



Open problem

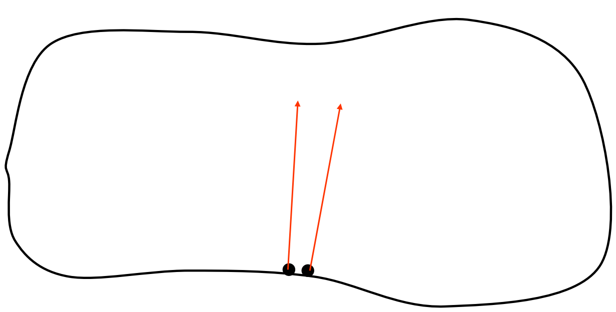
Open problem. Are there any bounded domains D with $\Lambda(D) < 0$?
Idea for an example – domain with many circular holes.



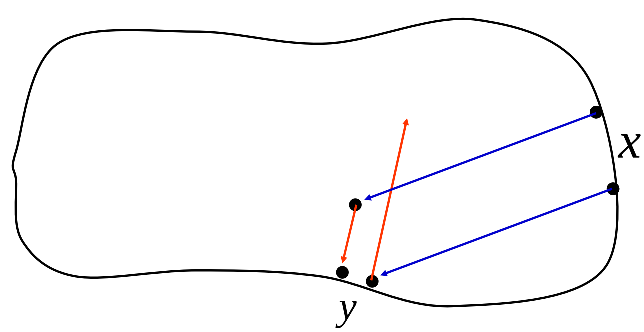
Lemma. If D is the exterior of a disc then $\Lambda(D) = 0$

Another idea – use ellipses instead of discs.
Lemma (Virag). If D is the exterior of an ellipse then $\Lambda(D) = 0$

$$\Lambda(D) = \int_{\partial D} \upsilon(x)dx + \iint_{\partial D \times \partial D} |\log\cos\alpha(x,y)|\, \chi_x(dy)dx$$



$$\Lambda(D) = \int_{\partial D} \upsilon(x)dx + \iint_{\partial D \times \partial D} |\log\cos\alpha(x,y)|\, \chi_x(dy)dx$$



Open problem

X_t^y - stochastic flow

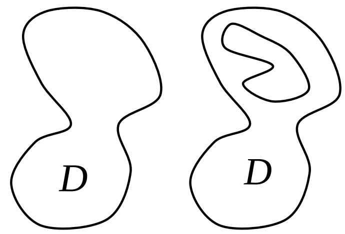
X_t^y starts from $y \in D$

All X_t^y are driven by the same Brownian motion B_t

Is it true that

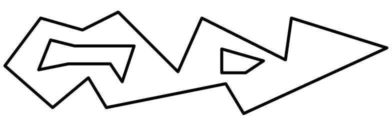
$$P\left(\lim_{t\to\infty} \sup_{x,y\in D} dist(X_t^x, X_t^y) = 0 \right) = 1?$$

Lions & Sznitman (1984)

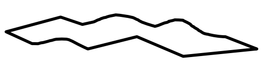


B and Chen (2002) : Synchronous couplings converge in

(i) Polygonal domains



(ii) Lip domains

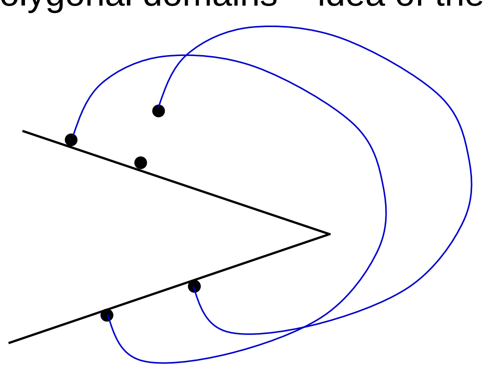


Open problem

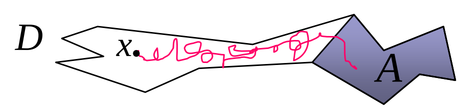
Does there exist a bounded planar domain in which synchronous couplings do not converge?

$$P(\limsup_{t\to\infty} dist(X_t, Y_t) > 0) > 0$$

Polygonal domains – idea of the proof



Heat equation and reflected Brownian motion



$u(x,t) =$ temperature at x at time t (Neumann boundary conditions)

X_t - reflected Brownian motion in D

$$u(x,0) = f(x) = 1_A(x)$$

$$u(x,t) = E^x f(X_t) = P^x(X_t \in A)$$

Neumann eigenfunctions

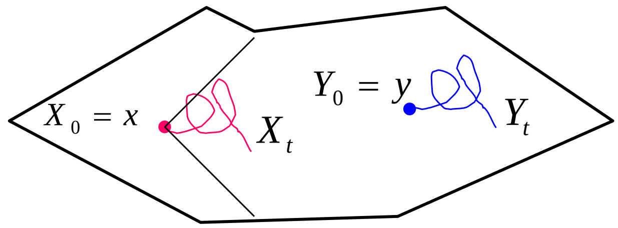
$$u(x,t) = c + \varphi(x)e^{-\lambda t} + \ldots$$

$$\varphi(x) - \varphi(y) \approx e^{\lambda t}(u(x,t) - u(y,t))$$

$$\varphi(x) - \varphi(y) \approx e^{\lambda t}(P^x(X_t \in A) - P^y(X_t \in A))$$

$$\varphi(x) - \varphi(y) \approx e^{\lambda t}(P^x(X_t \in A) - P^y(Y_t \in A))$$

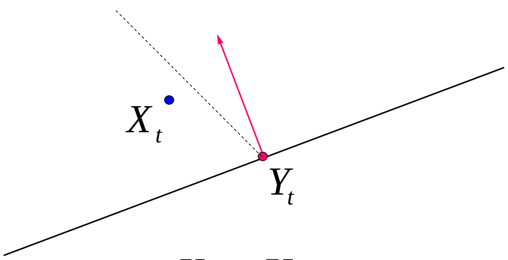
B and Kendall (2000)



$$x \le y \Rightarrow X_t \le Y_t \quad \forall t$$

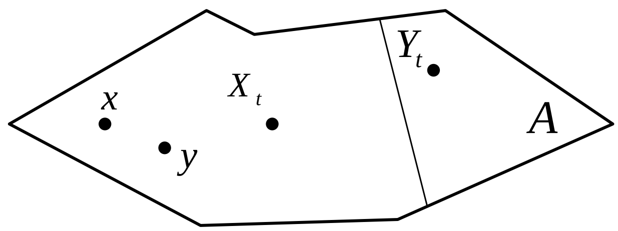
(monotonicity)

Effect of reflection



$$X_t \le Y_t$$

Eigenfunction monotonicity



$$x \le y \Rightarrow X_t \le Y_t \quad \forall t$$

$$P^x(X_t \in A) \le P^y(Y_t \in A)$$

$$\varphi(x) \le \varphi(y)$$

Synchronous couplings in lip domains

$$\varphi(X_t)e^{-\lambda t} - \varphi(Y_t)e^{-\lambda t}$$

is a non-negative martingale