# Perfectoid spaces and their applications

February 17-21, 2014

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1 Almost ring theory

Recall that there is a “tilting correspondence” relating objects in characteristic zero to objects in characteristic $p$. This correspondence works not at an integral level, but at an “almost integral level,” i.e. using the language of almost mathematics. A good general reference for almost mathematics is the book [GR03] of Gabber and Ramero.

1.1 Almost mathematics

Throughout, a non-archimedean field $K$ is a field equipped with a rank-one valuation $|·|: K^× → \mathbb{R}_{>0}$. If $K$ is any non-archimedean field, $K^\circ$ denotes the valuation ring $\{x \in K: |x| \leq 1\}$ of $K$.

Definition 1.1.1. A perfectoid field $K$ is a complete non-archimedean field such that:

1. The residue characteristic of $K$ is $p > 0$.
2. The associated rank-one valuation is non-discrete.
3. The Frobenius map $\Phi: K^\circ/p → K^\circ/p$ is surjective.

Example 1.1.2. The $p$-adic completion of $\mathbb{Q}_p(p^{1/p^\infty})$ is perfectoid.

Example 1.1.3. The $p$-adic completion $\mathbb{C}_p$ of $\mathbb{Q}_p$ is perfectoid.

Example 1.1.4. If $k$ is a non-archimedean field of positive characteristic, then the “perfectification” of $k$ is perfectoid. Here the perfectification of $k$ is the completion of the colimit $\varprojlim_{\Phi} k$, where $\Phi$ is Frobenius. The perfectification of $\mathbb{F}_p((t))$ is the $t$-adic completion of $\mathbb{F}_p((t))(t^{1/p^\infty})$.

Example 1.1.5. The field $\mathbb{Q}_p$ is not perfectoid because its valuation is discrete.

Let $K$ be a perfectoid field, and denote by $m \subset K^\circ$ the unique maximal ideal. Since the valuation on $K$ is non-discrete, $m$ is not finitely generated. One has $m \otimes m → m = m^2$, the first isomorphism coming from the flatness of $m$. Let $\Sigma \subset K^\circ\text{-Mod}$ be the full subcategory consisting of $m$-torsion modules. Since $m^2 = m$, $\Sigma$ is a (thick) abelian Serre subcategory.

Definition 1.1.6. Let $K$ be a perfectoid field.

1. A $K^\circ\text{-module}$ $M$ is almost zero if $M \in \Sigma$, i.e. $mM = 0$.
2. The category of $K^{\circ a}\text{-modules}$ is the Serre quotient $K^\circ\text{-Mod}/\Sigma$.

For example, the residue field $K^\circ/m$ is almost zero. On the other hand, $K^\circ/p$ (or $K^\circ/t$ for any $t \in m$) is not almost zero.

Write $(-)^a$ for the localization functor $K^\circ\text{-Mod} → K^{\circ a}\text{-Mod}$. A crucial fact is that $(-)^a$ has both left and right adjoints, denoted $N → N_!$ and $N \mapsto N_*$, respectively. This allows us to easily compute hom-sets in $K^{\circ a}\text{-Mod}$. Indeed,

\[
\text{hom}_{K^\circ}(M_!, N) = \text{hom}_{K^{\circ a}}(M^a, N^a) = \text{hom}_{K^\circ}(M, N_*)
\]
so the problem of computing $\hom_{K^{\text{op}}}(M^a,N^a)$ is reduced to that of computing $M_l$ and $N_s$. Fortunately, the functors $(-)_!$ and $(-)_*$ can be explicitly described. One has

$$\begin{align*}
(T^a)_! &= m \otimes T \\
(T^a)_* &= \hom_{K^{\text{op}}}(m,T).
\end{align*}$$

For a $K^{\text{op}}$-module $M$, one calls $M_s$ the module of “almost elements” of $M$.

This notation is motivated by topology. If $j : U \rightarrow X$ is the inclusion of an open subset into a topological space, then the restriction functor $j^* : \text{Sh}(X) \rightarrow \text{Sh}(U)$ has left and right adjoints $j_!$ and $j_*$. This suggests that $K^{\text{op}}\text{-Mod}$ is the category of sheaves on some subscheme of $\text{Spec}(K^a)$ that lies “in between” the special point and generic fiber, even though no such subscheme exists.

It is an easy exercise (and purely formal) to prove that $N \rightarrow N_l$ is an exact functor. Moreover, if $N$ is a $K^{\text{op}}$-module, then $(N_l)^a = N = (N_s)^a$, from which we see that $N \rightarrow N_l$ and $N \rightarrow N_s$ are fully faithful. (This too is an exercise in pure category theory.)

While $(-)_!$ and $(-)_*$ are sections of $(-)^a$, one does not generally have $(M^a)_! = M$ or $(M^a)_* = M$, for $M$ a $K^{\text{op}}$-module. For example, if $M = K^a$, then $(M^a)_! = m \neq M$. Similarly, if $M = m$, then $(M^a)_* = K^a \neq M$.

The subcategory $\Sigma \subset K^{\text{op}}\text{-Mod}$ is an ideal in the sense that it is closed under taking tensor products with arbitrary $K^{\text{op}}$-modules. Thus $K^{\text{op}}\text{-Mod}$ inherits a tensor product from $K^{\text{op}}\text{-Mod}$, so we can talk about algebras in $K^{\text{op}}\text{-Mod}$. For example, any $K^{\text{op}}$-algebra $A$ induces a $K^{\text{op}}$-algebra $A^a$. Using the functor $(-)_*$, one can show that every $K^{\text{op}}$-algebra is of this form. Moreover, with the obvious definition of a module over a $K^{\text{op}}$-algebra $A^a$, one sees that all $A^a$-modules are of the form $M^a$, for $M$ an $A$-module.

The abelian tensor category $(K^{\text{op}}\text{-Mod}, \otimes)$ has an internal hom-functor. For $K^{\text{op}}$-modules $M, N$, the hom-set $\hom_{K^{\text{op}}}(M,N)$ is naturally a $K^{\text{op}}$-modules, so we can put

$$\text{alHom}(M,N) = \hom_{K^{\text{op}}}(M,N)^a.$$  

1.2 Almost commutative algebra

As before, let $K$ be a perfectoid field.

**Definition 1.2.1.** Let $A$ be a $K^{\text{op}}$-algebra, $M$ an $A$-module.

1. $M$ is flat if $M \otimes_A -$ is an exact functor.
2. $M$ is almost projective if the functor $\text{alHom}(M,-)$ is exact.
3. Assume $A = R^a$ and $M = N^a$. Then $M$ is almost finitely generated if for all $\epsilon \in m$, there exists a finitely generated $R$-module $N_\epsilon$ with a map $f_\epsilon : N_\epsilon \rightarrow M$ such that $\ker(f_\epsilon)$ and $\coker(f_\epsilon)$ are killed by $\epsilon$.
4. If the number of generators of the $N_\epsilon$ can be taken to be bounded, we say that $M$ is uniformly finitely generated.

There is an obvious notion of an almost finitely presented $A$-module along the same lines.

Let $A = R^a$ be an almost $K^{\text{op}}$-algebra, and suppose $M = N^a$. Then $M$ is almost flat if and only if $\text{Tor}_i^R(N,-)$ takes values in almost zero modules for all $i > 0$. Similarly, $M$ is almost projective if and only if $\text{Ext}_R^i(M,-)$ takes values in almost zero modules for all $i > 0$. 

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It is not the case that an almost $A$-module is almost projective if and only if it is projective in the categorical sense. In fact, it is a good exercise to show that if $M$ is a $K^{\text{op}}$-module that is projective, then $M = 0$.

Any finitely generated ideal $I \subset K^{\text{op}}$ is an almost finitely generated $K^{\text{op}}$-module. In fact, such $I$ are uniformly almost finitely generated. Fix $r \in \mathbb{R}_{>0}$ such that $r \not\in |K^X|$. Then the ideal $I_r = \{ f \in K^{\text{op}} : |f| < r \}$ is not finitely generated, but is uniformly almost finitely generated.

1.3 Unramified and étale morphisms

Suppose $A \to B$ is a finite étale map of commutative rings. Then the diagonal $\text{Spec}(B) \to \text{Spec}(B \otimes_A B)$ is clopen. It follows that there is a unique idempotent $e \in B \otimes_A B$, called the diagonal idempotent such that

1. $e^2 = e$
2. $\mu(e) = 1$, where $\mu : B \otimes_A B \to B$ is the multiplication map
3. $\ker(\mu) \cdot e = 0$.

For example, if $A \to B$ is Galois with group $G$, then $B \otimes_A B \simeq \prod_{g \in G} B$ via $(b_1 \otimes b_2) \mapsto (b_1 \cdot g(b_2))_{g \in G}$. In this setting, $e$ is the element $(1, 0, \ldots, 0)$.

If one writes $e = \sum_{i=1}^N x_i \otimes y_i$ with $x_i, y_i \in B$, then $\text{tr}(e) = \sum \text{tr}(x_i y_i) = 1$. Moreover, consider the maps $\alpha : B \to A^B$ and $\beta : A^B \to B$ defined by

$$\alpha(b) = (\text{tr}(bx_i))_i,$$
$$\beta(a_i)_i = \sum_i a_i y_i.$$

One has $\beta \circ \alpha = 1_B$, so $B$ is projective.

1.4 Almost étale extensions

As before, let $K$ be a perfectoid field.

**Definition 1.4.1.** Let $f : A \to B$ be a morphism of $K^{\text{op}}$-algebras.

1. $f$ is unramified if there exists $e \in (B \otimes_A B)_*$ such that $e^2 = e$, $\mu(e) = 1$, and $\ker(\mu) \cdot e = 0$.
2. $f$ is étale if it is flat and unramified.
3. $f$ is finite étale if it is étale and $B$ is an almost finitely presented projective $A$-module.

If $A$ is a $K^{\text{op}}$-algebra, write $A_{\text{fet}}$ for the category of finite étale $A$-algebras. There is a good deformation theory for finite étale extensions. For example, if $I \subset A$ is nilpotent, then the natural functor $A_{\text{fet}} \to (A/I)_{\text{fet}}$ is an equivalence of categories.

Suppose $K$ is a perfectoid field of characteristic $p > 0$. Choose an element $0 \neq t \in \mathfrak{m}$. Let $A$ be a flat $K^{\text{op}}$-algebra which is integrally closed inside $A[1/t]$. Let $B'$ be a finite étale $A[1/t]$-algebra. Let $B$ be the integral closure of $A$ in $B'$.

**Proposition 1.4.2.** If $A$ is perfect, then $A^{\text{op}} \to B^{\text{op}}$ is finite étale as a map of $K^{\text{op}}$-algebras.
Proof. We have a diagonal idempotent $e \in B' \otimes_A B'$. There exists some $N > 0$ such that $t^N \cdot e \in B \otimes A B$. Everything in sight is perfect, so we can apply the inverse of Frobenius to conclude that $(t^N)^{1/p^n} \cdot e \in B \otimes A B$ for all $n > 0$. Thus $e \in (B \otimes A B)$ is almost integral, so $B$ is almost étale.

To see that $B$ is almost finitely presented, fix $\epsilon \in \mathfrak{m}$. Write $\epsilon \cdot e = \sum_{i=1}^N x_i \otimes y_i$. The composite $B \to A^{\oplus N} \to B$ of $b \mapsto (\text{tr}(bx_i))_i$ and $(a_i) \mapsto \sum a_i y_i$ is multiplication by $\epsilon$, so $B$ is uniformly almost finitely presented projective. \qed
2 Adic spaces

The main goal is to introduce the category of adic spaces over a non-archimedean base field, which generalizes rigid geometry. We want this framework to be able to handle more general affinoid rings (no finiteness conditions). The structure sheaf should be defined on all opens, not just with respect to a Grothendieck topology.

2.1 Affinoid algebras

Fix a non-archimedean base field $k$, i.e. $k$ is a complete topological field, with the topology defined by a nontrivial rank-one valuation $|\cdot| : k^\times \to \mathbb{R}_{\geq 0}$. Main examples include $\mathbb{Q}_p$, $\mathbb{F}_q((t))$, and any perfectoid field.

**Definition 2.1.1.** An $f$-adic ring is a topological ring $R$ such that $R$ contains an open subring $R_0$ (called the ring of definition) such that the topology on $R_0$ is adic with respect to a finitely generated ideal of definition.

**Definition 2.1.2.** An $f$-adic ring $R$ is called a Tate ring if there exists a topologically nilpotent unit in $R_0$.

If $R$ is a topological $k$-algebra, then $R$ is a Tate ring if the sets $\{aR_0 : a \in k^\times\}$ form a basis for the topology on $R$.

**Example 2.1.3.** Consider the ring $k\langle T_1, \ldots, T_n \rangle = k^\circ[T_1, \ldots, T_n][\frac{1}{\pi}]$, where $(-)^\pi$ denotes $\pi$-adic completion for some $\pi \in k^\times$ with $|\pi| < 1$.

An element $a \in R$ is called power-bounded if the set $\{a^n : n \geq 0\}$ is a bounded subset of $R$. In this context, a set $S \subset R$ is bounded if for any open neighborhood $U$ of 0, there exists $a \in k^\times$ such that $aU \supset S$. Write $R^\circ$ for the subring of $R$ consisting of power-bounded elements.

**Definition 2.1.4.** An affinoid algebra is a pair $(A, A^+)$ with $A$ an $f$-adic ring and $A^+ \subset A^\circ$ an open integrally closed subring.

We say that an affinoid algebra $(A, A^+)$ is of finite type over $k$ if $A$ is a quotient of $k(T_1, \ldots, T_n)$ for some $n$, and $A^+ = A^\circ$. From now on, let $A$ denote either an $f$-adic ring or a Tate ring.

2.2 Valuations

**Definition 2.2.1.** A valuation on a ring $A$ is a map $v : A \to \Gamma \cup \{0\}$, where $\Gamma$ is a totally ordered abelian group (written multiplicatively), such that

1. $v(0) = 0$ and $v(1) = 1$
2. $v(ab) = v(a)v(b)$
3. $v(a + b) \leq \max\{v(a), v(b)\}$

Eugene Hellman (Feb. 16)
Here as is typical, we put $\gamma > 0$ for all $\gamma \in \Gamma$, and define $\gamma \cdot 0 = 0 \cdot \gamma = 0$.

If $v$ is a valuation on $A$, let $\text{supp}(v) = \{ x \in A : v(x) = 0 \}$: this is an ideal in $A$. Let $\Gamma_v$ be the subgroup of $\Gamma$ generated by $\{ v(x) : x \in A, v(x) \neq 0 \}$. We say that two valuations $v, v'$ are equivalent if for all $a, b \in A$, one has $v(a) \leq v(b) \iff v'(a) \leq v'(b)$. Two valuations $v, v'$ are equivalent in this sense if and only if $\text{supp}(v) = \text{supp}(v')$ and the associated valuation rings in $\text{Frac}(A/\text{supp}(v))$ are the same.

Suppose $v$ is a valuation on a topological ring $A$. We call $v$ continuous if for all $\gamma \in \Gamma_v$, there exists an open neighborhood $U$ of zero in $A$ such that $v(x) < \gamma$ for all $x \in U$.

**Definition 2.2.2.** Let $A$ be an $f$-adic ring. Define $\text{Spv}(A)$ to be the set of equivalence classes of valuations on $A$.

**Definition 2.2.3.** If $A$ is an $f$-adic ring, $\text{Cont}(A)$ is the set of equivalences of continuous valuations on $A$.

**Definition 2.2.4.** Let $(A, A^+)$ be an affinoid ring. Define $\text{Spa}(A, A^+)$ to be the set of continuous valuations $v$ on $A$ such that for all $a \in A^+$, $v(a) \leq 1$.

We equip all the spaces with a topology generated by the sets $U_{f,g} = \{ v : v(f) \leq v(g) \neq 0 \}$ for $f, g \in A$.

**Theorem 2.2.5.** The spaces $\text{Spv}(A)$, $\text{Cont}(A)$ and $\text{Spa}(A, A^+)$ are spectral spaces.

### 2.3 Adic spectra

Let $(A, A^+)$ be an affinoid ring. For $x \in \text{Spa}(A, A^+)$ corresponding to a valuation $v$, and $f \in A$, put $|f(x)| = v(f)$.

If $A$ is a Tate $k$-algebra, then the rational subsets

$$U \left( \frac{f_1, \ldots, f_n}{g} \right) = \{ x \in \text{Spa}(A, A^+) : |f_i(x)| \leq |g(x)| \text{ for all } i \}$$

for $f_1, \ldots, f_n \in A$ with $(f_1, \ldots, f_n) = A$, form a basis for the topology on $\text{Spa}(A, A^+)$. 

**Example 2.3.1.** Let $A$ be a Tate $k$-algebra of finite type. If $m \subset A$ is a maximal ideal, then we get a valuation on $A$ by composing $A \rightarrow A/m$ with the unique extension of $|\cdot|_k$ to $A/m$. This gives a canonical inclusion

$$\text{Sp}(A) = \{ m \in \text{Spec}(A) : m \text{ maximal} \} \hookrightarrow \text{Spa}(A, A^c).$$

If $A$ is a finite-type Tate $k$-algebra, then we have a corresponding statement for the corresponding rigid analytic space associated to $A$. The topology on $\text{Spa}(A, A^c)$ induces the standard Grothendieck topology on the standard rigid-analytic space. One has bijections

$$\{ \text{q-c opens in } \text{Spa}(A, A^c) \} \leftrightarrow \{ \text{q-c opens in } \text{Sp}(A) \}$$

$$\{ \text{coverings by q-c opens} \} \leftrightarrow \{ \text{admissible covers by q-c admissible opens} \}$$
Theorem 2.3.4. If $\mathcal{O}$ is any ring of definition. Let $A$ be as above. Then $\mathcal{O}(\mathcal{U})$ is a sheaf on $X = \text{Spa}(A, A^+)$ up to five different types of points, depending on whether $k$ is spherically complete. These are the “classical points” coming from rigid geometry. Every element $x \in k^0$ induces a valuation $f \mapsto |f(x)|$.

The next two types of points are constructed in the same way. Let $x \in k^0$, $r \in \mathbb{R}_{>0}$. The valuation $v_{x,r}$ is defined by either of the following formulas:

$$v_{x,r} \left( \sum (T - x)^i \right) = \sup_{i \geq 0} |a_i| \cdot r^i$$

$$v_{x,r}(f) = \sup_{y \in D_r(x)} |f(y)|$$

where $D_r(x) = \{ y \in k^0 : |x - y| \leq r \}$ is the closed ball of radius $r$ about $x$.

1. If $r \in [k^x]$, we say that $v_{x,r} \in X$ is type 1.
2. If $r \notin [k^x]$, we say that $v_{x,r} \in X$ is type 2.
3. If $r \notin [k^x]$, we say that $v_{x,r} \in X$ is type 3.
4. Suppose $k$ is not spherically complete, i.e. there exists a sequence $D_\star = D_1 \supset D_2 \supset \cdots$ of open balls such that $\bigcap_i D_i = \emptyset$. The valuation

$$v_{D_\star}(f) = \inf_{i \geq 1} \sup_{y \in D_i} |f(y)|$$

We’d like to construct a structure sheaf on $X = \text{Spa}(A, A^+)$. For simplicity’s sake, we assume $(A, A^+)$ is Tate. Let $U = U \left( \frac{\mathcal{O}_g}{\mathcal{O}_g} \right) \subset X$ be a standard rational subset; we need to define $\mathcal{O}_X(U)$. Let $A(\frac{\mathcal{O}_g}{\mathcal{O}_g})$ be the completion of $A[\frac{\mathcal{O}_g}{\mathcal{O}_g}]$ with respect to the topology generated by opens of the form $a \cdot A_0[\frac{\mathcal{O}_g}{\mathcal{O}_g}]$, for $a \in k^\times$. (Here $A_0 \subset A$ is any ring of definition.) Let $A(\frac{\mathcal{O}_g}{\mathcal{O}_g})$ be the completion of the integral closure of $A[\frac{\mathcal{O}_g}{\mathcal{O}_g}]$ with respect to the same topology.

**Proposition 2.3.2.** The canonical map $\text{Spa}(A(\frac{\mathcal{O}_g}{\mathcal{O}_g}), A(\frac{\mathcal{O}_g}{\mathcal{O}_g})) \to (\text{Spa} A, A^+)$ induced by $\varphi: (A, A^+) \to (A(\frac{\mathcal{O}_g}{\mathcal{O}_g}), A(\frac{\mathcal{O}_g}{\mathcal{O}_g})^+)$ factors over $U$. Moreover, $\varphi$ is universal for maps $(A, A^+) \to (B, B^+)$ with $B$ complete such that $\text{Spa}(B, B^+) \to \text{Spa}(A, A^+)$ factors over $U$.

For $U$ as above, we define $\mathcal{O}_X(U) = A(\frac{\mathcal{O}_g}{\mathcal{O}_g})$ and $\mathcal{O}_X^+(U) = A(\frac{\mathcal{O}_g}{\mathcal{O}_g})^+$. For general $U$, we put

$$\mathcal{O}_X(U) = \lim_{\mathcal{W} \subseteq U} \mathcal{O}_X(W).$$

It is not known whether $\mathcal{O}_X$ as defined is a sheaf in general. This is easily circumvented by the following definition.

**Definition 2.3.3.** Let $(A, A^+)$ be as above. Then $\text{Spa}(A, A^+)$ is called an affinoid adic space with sheaf $\mathcal{O}_X$ if the presheaf $\mathcal{O}_X$ is a sheaf.

**Theorem 2.3.4.** If $A$ is strongly noetherian (i.e. $A(T_1, \ldots, T_n)$ is noetherian for all $n \geq 0$) then $\mathcal{O}_X$ is a sheaf on $X = \text{Spa}(A, A^+)$. For example, Tate algebras of finite type over any non-archimedean field $k$ are strongly noetherian.

**Example 2.3.5** (The closed unit ball). Let $k$ be a non-archimedean field which is complete and algebraically closed, for example $\mathbb{C}_p$. Let $X = \text{Spa}(k(T), k^0(T))$. The space $X$ can have up to five different types of points, depending on whether $k$ is spherically complete.

1. These are the “classical points” coming from rigid geometry. Every element $x \in k^0$ induces a valuation $f \mapsto |f(x)|$.

2. If $r \in [k^x]$, we say that $v_{x,r} \in X$ is type 2.

3. If $r \notin [k^x]$, we say that $v_{x,r} \in X$ is type 3.

4. Suppose $k$ is not spherically complete, i.e. there exists a sequence $D_\star = D_1 \supset D_2 \supset \cdots$ of open balls such that $\bigcap_i D_i = \emptyset$. The valuation

$$v_{D_\star}(f) = \inf_{i \geq 1} \sup_{y \in D_i} |f(y)|$$

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is said to be of type 4.

5. Points of type 1, 3, and 4 are closed and rank one. Points of type 2 are not closed (but have rank one) – they have (rank two) specializations of the following type, illustrated by an example. The point \( v_{x,1} \) has specialization \( \xi_{x,1} \), which takes values in \( \mathbb{R}_{>0} \times \langle \gamma \rangle \), where we put \( \gamma < \mathbb{R}_{>0} \) (i.e. use the lexicographic order). Put

\[
\xi_{x,1}(f) = \sup_{i \geq 1} |a_i| \cdot \gamma^i \in \mathbb{R}_{>0} \times \gamma^\mathbb{Z}.
\]

### 2.4 Glueing of affinoid adic spaces

One glues adic spaces in the category \( \mathcal{V} \) consisting of triples \( (X, \mathcal{O}_X, \{v_x : x \in X\}) \), where \( (X, \mathcal{O}_X) \) is a locally ringed space, \( \mathcal{O}_X \) is a sheaf of complete topological rings on \( X \), and for each \( x \in X \), \( v_x \) is a valuation on \( \mathcal{O}_{X,x} \). Morphisms in \( \mathcal{V} \) are required to respect all the data.

If \( (A, A^+) \) is an affinoid ring such that \( \text{Spa}(A, A^+) \) is an adic space, then for \( x \in X = \text{Spa}(A, A^+) \), there is a canonical extension of the valuation \( v_x \) to \( \mathcal{O}_{X,x} \).

It is necessary to keep track of the valuations \( \{v_x : x \in X\} \) to be able to recover \( \mathcal{O}_X^+ \). The corresponding category of triples \( (X, \mathcal{O}_X, \mathcal{O}_X^+) \) is equivalent to \( \mathcal{V} \).

**Definition 2.4.1.** An adic space is an object of \( \mathcal{V} \) that is locally isomorphic to an affinoid adic space.
3 Almost ring theory 2: perfectoid rings

Here we treat the affine case of the tilting correspondence for perfectoid spaces. Unless anything is said to the contrary, all theorems are due to Scholze. If $R$ is a ring of characteristic $p$, let $\Phi : R \to R$ be the Frobenius $x \mapsto x^p$.

3.1 Tilting for fields

Let $K$ be a perfectoid field.

Definition 3.1.1. The tilt of $K$ has valuation ring $K^{\flat} = \lim_{\leftarrow} K^o/p$. So $K^\flat = \text{Frac}(K^o)$.

Theorem 3.1.2 (Fontaine, Wintenberger). If $K$ is perfectoid, so is $K^\flat$, and tilting induces an equivalence of categories $K_{\text{f\acute{e}t}} \simeq K^\flat_{\text{f\acute{e}t}}$.

One proves this by starting with $K$, passing to the integral level via $K^o$, and noting that $K^{\flat} \to K^o/\pi \simeq K^o/p$ for some $\pi$ with $|p| \leq |\pi| < 1$. Our goal is to generalize this by defining categories of perfectoid algebras over $K$ and $K^\flat$, and proving that these categories are equivalent via a generalized tilting functor.

It is not so obvious, but every perfectoid field of characteristic $p > 0$ is the tilt of a perfectoid field of characteristic zero.

3.2 Tilting more general algebras

Let $K$ be a perfectoid field, and choose some nonzero $\pi \in \mathfrak{m}$ with $|p| \leq |\pi| < 1$.

Definition 3.2.1. 1. A Banach $K$-algebra $R$ is perfectoid if $R^o \subset R$ is open and bounded, and $\Phi : R^o/p \to R^o/p$ is surjective. Let $K\text{-Perf}$ be the category of perfectoid $K$-algebras and continuous morphisms of $K$-algebras.

2. A perfectoid $K^o\$-algebra is a flat $K^{\text{o\$}}$-algebra $R$ that is $\pi$-adically complete and Frobenius induces an isomorphism $R/\pi^1/p \to R/\pi$. Let $(K^{\text{o\$}})$-Perf be the category of perfectoid algebras over $K^{\text{o\$}}$.

3. A perfectoid $(K^{\text{o\$}}/\pi)$-algebra $R$ is a flat $(K^{\text{o\$}}/\pi)$-algebra $R$ such that Frobenius induces an isomorphism $R/\pi^1/p \to R$. Let $(K^{\text{o\$}}/\pi)$-Perf be the category of perfectoid $(K^{\text{o\$}}/\pi)$-algebras.

Example 3.2.2. Let $R = K\langle T^1/p^\infty \rangle$ in the sense of Tate, i.e.

$$R = \left( \bigcup_{n \geq 1} K^o\langle T^1/p^n \rangle \right)^\wedge \left[\frac{1}{p}\right].$$

Example 3.2.3. Similarly, $A = (K^o\langle T^1/p^\infty \rangle)^a$ is a perfectoid $K^o$-algebra.

Example 3.2.4. If $K$ has characteristic $p > 0$, then perfectoid $K$-algebras are simply Banach $K$-algebras $A$ such that $A^o$ is open and bounded.
Theorem 3.2.5 (Tilting equivalence). There are natural equivalences of categories

\[ K \text{-} \text{Perf} \overset{(a)}{\sim} K^{oa} \text{-} \text{Perf} \overset{(b)}{\sim} (K^{oa}/\pi) \text{-} \text{Perf} \]

\[ K^{oa} \text{-} \text{Perf} \overset{(c)}{\sim} (K^{hoa}/\pi) \text{-} \text{Perf} \]

One proves equivalences (a) and (d) using almost mathematics, while (c) and (d) require some deformation theory – in particular the cotangent complex.

3.3 Comparing \( K \text{-} \text{Perf} \) and \( K^{oa} \text{-} \text{Perf} \)

Lemma 3.3.1. Let \( M \) be a \( K^{oa} \text{-} \text{module} \), \( N \) a \( K^{o} \text{-} \text{module} \). Then

1. \( M \) is flat in over \( K^{oa} \) if and only if \( M^{\pi} \) is flat over \( K^{o} \).
2. If \( N \) is flat over \( K^{o} \), then \( N^{a} \) is flat, and
   \[ N^{a} = \left\{ x \in N \left[ \frac{1}{\pi} \right] : \epsilon \cdot x \in N \text{ for all } \epsilon \in m \right\} \]
3. If \( M \) is flat over \( K^{oa} \), then \( M^{\pi} \) is \( \pi \)-adically complete if and only if \( M^{\pi} \) is \( \pi \)-adically complete.

Theorem 3.3.2. The functors

\[ K \text{-} \text{Perf} \to K^{oa} \text{-} \text{Perf} \quad R \mapsto (R^{o})^{a} \]
\[ K^{oa} \text{-} \text{Perf} \to K \text{-} \text{Perf} \quad A \mapsto A^{[\frac{1}{\pi}]} \]

are an equivalence of categories.

The proof of this theorem relies in part on the following proposition.

Proposition 3.3.3. If \( R \) is perfectoid over \( K \), then \( R^{oa} \) is perfectoid over \( K^{oa} \).

Proof. By assumption, \( \Phi : R^{o}/\pi \to R^{o}/\pi \) is surjective. We show that its kernel is \( \pi^{1/p} \cdot R^{o} \). Let \( x \in R^{o} \) such that \( x^{p} \in \pi^{1} \cdot R^{o} \). Then \( x^{p} = \pi \cdot y \) for some \( y \in R^{o} \). Thus \( (x/\pi^{1/p})^{p} = y \in R^{o} \), so \( x/\pi^{1/p} \in R^{o} \), whence \( x \in \pi^{1/p} \cdot R^{o} \).

To see that \( R^{oa} \) is \( \pi \)-adically complete, refer to Lemma 3.3.1. \( \square \)

Proposition 3.3.4. Let \( A \) be a perfectoid \( K^{oa} \text{-} \text{algebra} \). Define \( R = A_{s}^{[\frac{1}{\pi}]} \), with the Banach structure making \( A_{s} \) open and bounded. Then \( A_{s} = R^{o} \) and \( R \) is perfectoid.

Proof. First, note that \( \Phi : A_{s}/\pi^{1/p} \to A_{s}/\pi \) is injective by Lemma 3.3.1. Alternatively, we know that it is an almost isomorphism. So for \( x \in A_{s} \) with \( x^{p} \in \pi \cdot A_{s} \), we have \( \epsilon \cdot x \in \pi^{1/p} \cdot A_{s} \) for all \( \epsilon \in m \). It follows that \( x \in ((\pi^{1/p} \cdot A_{s})^{a})_{s} = \pi^{1/p} \cdot A_{s} \), whence \( \ker(\Phi) = 0 \).

Next, we assume \( x \in R \) has \( x^{p} \in A_{s} \) and show that \( x \in A_{s} \). Write \( y = \pi^{k/p} x \in A_{s} \) for some \( k \geq 1 \). Raise both sides to the \( p \) power, to obtain \( y^{p} = \pi^{k} x^{p} \in \pi A_{s} \). By the injectivity we have already proved, we get \( y \in \pi^{1/p} A_{s} \). Thus \( \pi^{(k-1)/p} x \in A_{s} \), so we can apply induction.
Finally, we show that $\Phi : A_*/\pi^{1/p} \to A_*/\pi$ is surjective. Because the map is already almost surjective, it is enough to show that the composite $A_*/\pi^{1/p} \to A_*/\pi \to A_*/\pi A_*$ is surjective. Choose $x \in A_*$. By almost surjectivity, we can write $\pi^{1/p} \cdot x \equiv y^p \mod \pi A_*$ for some $y$. Let $z = y/\pi^{1/p} \in R$. Then $z^p = y^p/\pi^{1/p} = x \mod \pi^{1-1/p}A_*$, so $z^p \in A_*$. We already have shown that this implies $z \in A_*$, so $z^p = q \mod \pi A_*$ implies that $x$ has a $p$-th root modulo $m$.

3.4 Some deformation theory

We'll start with a review of the cotangent complex as it appears in standard algebraic geometry. If $A \to B$ is a ring homomorphism, then the cotangent complex $L_{B/A}$ is an object in $D^\leq 0(B\text{-Mod})$. If $A \to B$ is smooth, then $L_{B/A} = \Omega^1_{B/A}[0]$. Most importantly, $L_{B/A}$ controls the deformation theory of $A \to B$ regardless of whether $B$ is smooth over $A$.

Example 3.4.1. Suppose $A = \mathbb{F}_p$ and $B$ is any perfect $\mathbb{F}_p$-algebra. It turns out that $L_{B/A} = 0$. This is easy to prove. By assumption, the Frobenius map $\Phi : B \to B$ is an isomorphism, so the induced morphism $d\Phi : L_{B/A} \to L_{B/A}$ is an isomorphism. But $d\Phi = 0$ because $d(y^p) = py^{p-1}dy = 0$.

Corollary 3.4.2. There exists a unique flat $(\mathbb{Z}/p^n)$-algebra $W_n(B)$ lifting $B$ from $\mathbb{F}_p$ to $\mathbb{Z}/p^n$.

One typically calls $W_n(B)$ the ring of $p$-typical Witt vectors in $B$ of length $n$.

3.5 Comparing $K^{oa}\text{-Perf}$ and $(K^{oa}/\pi)\text{-Perf}$

If $A \to B$ is a map of $K^{oa}$-algebras, then [GR03] define an object of $D(B\text{-Mod})$ that they call $L_{B/A}$. This $L_{B/A}$ is actually constructed as an “honest complex” in $D(B_*\text{-Mod})$. The complex $L_{B/A}$ satisfies all the properties one would hope for.

Lemma 3.5.1. Let $A$ be a perfectoid $(K^{oa}/\pi)$-algebra. Then $L_{A/(K^{oa}/\pi)} = 0$.

Proof. Essentially, this comes down to the fact that the relative Frobenius is an isomorphism. Consider the following commutative diagram:

$$
\begin{array}{c}
A \leftarrow \sim \quad A/\pi^{1/p} \leftarrow \quad A \\
\uparrow \quad \quad \quad \quad \quad \quad \downarrow \\
K^{oa}/\pi \leftarrow \sim \quad K^{oa}/\pi^{1/p} \leftarrow \quad K^{oa}/\pi
\end{array}
$$

The argument from Example 3.4.1 completes the proof.

Lemma 3.5.1 implies that $(K^{oa}/\pi)\text{-Perf} \simeq (K^{oa}/\pi^n)\text{-Perf} \simeq K^{oa}\text{-Perf}$. Explicitly, start with $A \in (K^{oa}/t)\text{-Perf}$. Then $A^p = \lim_{\leftarrow n} A$ is a perfectoid $K^{oa}$-algebra.
3.6 Tilting étale covers

Let $A$ be a perfectoid algebra over $K$. We have the following chain of equivalences of categories:

\[
A_{\text{fét}} \overset{(a)}{\sim} A_{\text{fét}}^{\text{oa}} \overset{(b)}{\sim} (A^{\text{oa}}/\pi)_{\text{fét}}
\]

\[
A_{\text{fét}}^{b} \overset{(d)}{\sim} A_{\text{fét}}^{\text{boa}} \overset{(c)}{\sim} (A^{\text{boa}}/\pi)_{\text{fét}}
\]

Essentially, we have seen that (b) and (d) follow from deformation theory, while (c) follows from the last example.
4 Adic spaces 2: perfectoid rings

4.1 Tilting perfectoid algebras

Fix a perfectoid base field $K$. This field contains the subring $K^\circ = \mathcal{O}_K$ of integral elements, which contains a unique maximal ideal $m$. Recall that the tilt of $K$ is

$$K^\flat \supset \mathcal{O}_K^\flat = \lim_{\leftarrow} \mathcal{O}_K/p \supset m^\flat$$

Choose a nonzero $\varpi \in m$, $\varpi^\flat \in m^\flat$ such that $\mathcal{O}_K/\varpi \simeq \mathcal{O}_K^\flat/\varpi^\flat$.

**Definition 4.1.1.** An affinoid $K$-algebra $(R,R^+)$ is perfectoid if $R$ is a perfectoid $K$-algebra.

Note that $m \subset R^\circ \subset R^+ \subset R^\circ$, and $R^+ \to R^\circ$ is an almost isomorphism. Also, the rings $R^+, R^\circ$ carry the $\varpi$-adic topology, and are complete with respect to this topology. (This is true because $R^\circ \subset R$ is bounded.)

**Proposition 4.1.2.** There is a tilting equivalence

$$\{(R,R^+) \text{ perfectoid } K\text{-algebras}\} \leftrightarrow \{(S,S^+) \text{ perfectoid } K^\flat\text{-algebras}\}$$

where a perfectoid $K$-algebra $R$ is mapped to $R^\flat = \lim_{x \mapsto x^p} R$ as a multiplicative monoid. One has $R^\circ \supset R^+ = \lim_{x \mapsto x^p} R^+$. There are natural isomorphisms $R^+/\varpi \simeq R^\flat/\varpi^\flat$.

We have a diagram

$$\begin{array}{ccc}
K & \longrightarrow & \mathcal{O}_K \\
\downarrow & & \downarrow \\
K^\flat & \longrightarrow & \mathcal{O}_K^\flat/\varpi^\flat
\end{array}$$

From Bhargav’s talk, we know that $R^\circ = \varprojlim_{\Phi} \mathcal{O}_K/\varpi$, and there is an obvious multiplicative projection

$$\varprojlim_{\Phi} R^\circ \to \varprojlim_{\Phi} R^\circ/\varpi.$$ 

It turns out that this is an isomorphism, so we can use it to transfer the additive structure of $\varprojlim_{\Phi} R^\circ/\varpi$ to $\varprojlim_{x \mapsto x^p} R^\circ$. For an arbitrary sequence $(\bar{x}_0, \bar{x}_1, \ldots) \in \varprojlim_{\Phi} R^\circ/\varpi$, choose lifts $\tilde{x}_i$ of $\bar{x}_i$ to $R^\circ$, and put

$$x_i = \lim_{n \to \infty} (\tilde{x}_{i+n})^p \in R^\circ$$

It is not hard to show that this gives a well-defined continuous multiplicative homomorphism

$$R^\flat = \varprojlim_{x \mapsto x^p} R \to R \quad f \mapsto f^\flat.$$

---

Peter Scholze (Feb. 18)
Proposition 4.1.3. There exists a continuous map $\text{Spa}(R, R^+) \to \text{Spa}(R^p, R^{p+})$, $x \mapsto x^p$

defined by

$$|f(x^p)| = |f^p(x)|$$

for $f \in R^p$.

Proof. First, we need to check that $x^p$ is actually a valuation. All the properties but the
triangle inequality are clear. But $f \mapsto f^p$ is not additive, so it is not a priori clear that
$x^p$ satisfies the triangle inequality. Let $f, g \in R^p$. Rescaling by an element of $K$, we may
assume that $f, g \in R^p$. Note that $f^p \equiv g^p \pmod{\varpi}$, as elements of $R^p/\varpi = R^{p+}/\varpi^p$. It follows that $f^p + g^p \equiv (f + g)^p \pmod{\varpi}$, i.e. $(-)^p$ is additive modulo $\varpi$. We can now compute

$$|(f + g)(x^p)|^{1/p^n} = |(f^{1/p^n} + g^{1/p^n})^p(x)|^{1/p^n}$$

$$\leq \max(|\pi|, |(f^{1/p^n} + (g^{1/p^n})^p(x)|)$$

$$\leq \max(|\pi|, |f^p(x)|^{1/p^n}, |g^p(x)|^{1/p^n})$$

$$= \max(|f^p(x)|, |g^p(x)|)^{1/p^n},$$

for $n \gg 0$. Raising to the $p^n$-th power, we get $|(f + g)(x^p)| = \max(|f(x^p)|, |g(x^p)|)$, as desired.

Theorem 4.1.4. Let $(R, R^+)$ be a perfectoid affinoid $K$-algebra with tilt $(R^p, R^{p+})$. Then

1. The map $(-)^p : X = \text{Spa}(R, R^+) \to \text{Spa}(R^p, R^{p+}) = X^p$ is a homeomorphism, preserving rational subsets.

2. The structure presheaves $\mathcal{O}_X$, $\mathcal{O}_X^+, \mathcal{O}_{X^p}$, are sheaves.

3. For all $U \subset X$ rational, $(\mathcal{O}_X(U), \mathcal{O}_X^+((U))$ is a perfectoid affinoid $K$-algebra with tilt $(\mathcal{O}_{X^p}(U), \mathcal{O}_{X^p}^+(U))$.

4. For all $x \in X$, the completed residue field $\hat{k}(x)$ is perfectoid with tilt $\hat{k}^p(x)$.

We have already seen that if $R$ is strongly noetherian, then the structure presheaves
on $\text{Spa}(R, R^+)$ are sheaves. In this theorem, there are no finiteness conditions – it is the
"perfectoidness" of $R$ that is used in showing that $\mathcal{O}_X$ etc. are sheaves.

The proof is based on an approximation lemma: given $f \in R$, we want there to exist
$g \in R^p$ such that $f^p - g^p$ is small. In some sense, this happens, but see Caraiani’s talk for more
details.

First we show that $\mathcal{O}_X$ is a sheaf if $K$ has characteristic $p$.

Definition 4.1.5. We say that $(R, R^+)$ is $p$-finite if there is a reduced Tate $K$-algebra
$(S, S^+)$, topologically of finite type, such that $R^+ = \varpi$-adic completion of $\lim_{\Phi} S^+$, and
$R = R^+[\frac{1}{\varpi}]$.

Proposition 4.1.6. In this situation, the map $X = \text{Spa}(R, R^+) \to Y = \text{Spa}(S, S^+)$
is a homeomorphism preserving rational subsets. Moreover, for any $U \subset X$ rational,
$(\mathcal{O}_X(U), \mathcal{O}_X^+((U))$ is $p$-finite, and is the perfect completion of $\mathcal{O}_Y(U), \mathcal{O}_Y^+(U))$.

Corollary 4.1.7. Let $X$ be as in the proposition. Let $X = \bigcup U_i$ be a finite cover by rational
$U_i \subset X$. Then

$$0 \longrightarrow R^+ \longrightarrow \prod \mathcal{O}_X^+(U_i) \longrightarrow \prod \mathcal{O}_X^+(U_{ij}) \longrightarrow \cdots$$

is almost exact (i.e. its cohomology is killed by $m$).
Proof. We already know that $\mathcal{O}_Y$ is a sheaf, so

$$0 \longrightarrow S \longrightarrow \prod \mathcal{O}_Y(U_i) \longrightarrow \prod \mathcal{O}_Y(U_{ij}) \longrightarrow \cdots$$

is exact. We can apply Banach’s open mapping theorem to show that in the complex

$$0 \longrightarrow S^+ \longrightarrow \prod \mathcal{O}_Y^+(U_i) \longrightarrow \prod \mathcal{O}_Y^+(U_{ij}) \longrightarrow \cdots$$

all the cohomology groups are killed by a (single) power of $\varpi$. Take the direct limit over Frobenius, all cohomology groups are almost zero. Complete everything in sight, and we get the result. \qed

As in Bhargav’s talks, the main idea is that properties true for the generic fiber are “almost true” (i.e. true in the almost context).

**Proposition 4.1.8.** Suppose $K$ is a perfectoid field of characteristic $p$. Any perfectoid affinoid $K$-algebra $(R, R^+)$ is a completed filtered direct limit of $p$-finite ones.

This is a generalization of the fact that any $\mathbb{Z}$-algebra is a direct filtered direct limit of $\mathbb{Z}$-algebras of finite type, and the proof runs similarly.

Note that a filtered direct limit of almost exact sequences is almost exact. This gives us the following corollary.

**Corollary 4.1.9.** For any $X = \text{Spa}(R, R^+)$, where $(R, R^+)$ is a perfectoid affinoid $K$-algebra and $K$ has characteristic $p$, the structure presheaves $(\mathcal{O}_X, \mathcal{O}_X^+)$ are sheaves.

### 4.2 Almost purity

Fix a perfectoid $K$-algebra $R$. In Bhargav’s talk we had the following diagram:

$$\begin{array}{ccc}
R_{\text{f\'{e}t}} & \xrightarrow{\sim} & R_{\text{f\'{e}t}}^{\text{ao}} \\
\downarrow & & \downarrow \\
R_{\text{f\'{e}t}}^\flat & \xrightarrow{\sim} & (R^\flat/\varpi)^{\text{f\'{e}t}} \\
\end{array}$$

We get a functor $R_{\text{f\'{e}t}}^\flat \hookrightarrow R_{\flat}$, inverse to the tilting functor.

**Theorem 4.2.1.** There is a natural equivalence $R_{\text{f\'{e}t}}^{\text{ao}} \simeq R_{\text{f\'{e}t}}$. Equivalently, for all finite étale $R$-algebras $S$, $S$ is perfectoid, and $S^{\text{ao}}$ is finite étale over $R^{\text{ao}}$.

This is the almost purity theorem of Faltings, motivated by the classical Zariski-Nagata purity theorem.

Take any such $S$. Choose $R^+ = R^\circ$, and let $X = \text{Spa}(R, R^+)$, and $X^\flat = \text{Spa}(R^\flat, R^{\flat+})$. If $U \subset X$ is rational, we have $S(U) = S \otimes_R \mathcal{O}_X(U)$, a finite étale $\mathcal{O}_X(U)$-algebra.

**Lemma 4.2.2.** Fix a point $x \in X$. Then

$$\varprojlim_{U \ni x} \mathcal{O}_X(U)_{\text{f\'{e}t}} \simeq \hat{\mathbb{k}}(x)_{\text{f\'{e}t}}.$$
Proof. We compute
\[
\lim_{\to} \left( \mathcal{O}_X(U) \right)_{\text{ét}} \cong \left( \lim_{\to} \mathcal{O}_X(U) \right)_{\text{ét}}.
\]

But \( \lim_{\to} \mathcal{O}_X^+(U) \) is henselian along \( \varpi \) because each \( \mathcal{O}_X^+(U) \) is complete. General theory tells us that
\[
(\mathcal{O}_{X,x})_{\text{ét}} \cong \left( \widehat{\mathcal{O}_{X,x}^+ \left[ \frac{1}{\varpi} \right]} \right)_{\text{ét}}.
\]

But in the exact sequence
\[
0 \longrightarrow I \longrightarrow \mathcal{O}_{X,x}^+ \longrightarrow k(x)^+ \longrightarrow 0
\]
the ideal \( I \) is a \( K \)-vector space. Indeed, if \( f \in I \), then \( |f| \leq |\varpi| \), so \( f \) lies in an open neighborhood of \( x \). Thus \( \frac{1}{\varpi} \in \lim_{\to} (\mathcal{O}_{X,x}^+ \to k(x)^+) = I \). Take \( \varpi \)-adic completion, and \( I \) vanishes, so \( \widehat{\mathcal{O}_{X,x}^+} = k(x)^+ \). It follows that \( \left( \widehat{\mathcal{O}_{X,x}^+ \left[ \frac{1}{\varpi} \right]} \right)_{\text{ét}} = \widehat{k(x)^+}_{\text{ét}} \).

**Corollary 4.2.3.**
\[
\lim_{\to} \mathcal{O}_X(U)_{\text{ét}} \cong k(x)_{\text{ét}} \cong \widehat{k(x^{\flat})}_{\text{ét}} \cong \lim_{\to} \mathcal{O}_X(U)_{\text{ét}}^{\flat}
\]

It follows that locally, \( S(U) \) is in the image of \( \mathcal{O}_X^+(U)_{\text{ét}} \hookrightarrow \mathcal{O}_X(U)_{\text{ét}} \). A basic gluing argument yields the result.

### 4.3 Perfectoid spaces

**Definition 4.3.1.** A perfectoid space over \( K \) is an adic space over \( K \) that is locally isomorphic to \( \text{Spa}(R, R^+) \), where \( (R, R^+) \) is a perfectoid affinoid \( K \)-algebra.

**Corollary 4.3.2.** There is an equivalence of categories
\[
\{ \text{perfectoid spaces over } K \} \simeq \{ \text{perfectoid spaces over } K^\flat \}
\]
given by \( X \mapsto X^\flat \). Moreover, \( |X| \simeq |X^\flat| \), \( \mathcal{O}_X \) tilts to \( \mathcal{O}_X^{\flat} \), when evaluated on affinoid perfectoid opens \( U \subset X \).
5 Introduction to $p$-adic comparison theorems

Unfortunately, titles of talks were not given by the speakers. So this talk will contain nothing about $p$-adic comparison theorems. A better title would be “constructing universal rings.”

5.1 Rings of Witt vectors

Fix a prime number $p > 0$. Cuntz and Denninger discovered a very beautiful way of constructing the ring of Witt vectors $W(R)$, for $R$ a perfect $F_p$-algebra. It involves some “naive multiplicative deformation theory.” For more details, see their preprint [CD13].

Let $\Lambda$ be a commutative ring, $R$ a $\Lambda$-algebra $n > 0$ an integer. Recall that a $\Lambda$-infinitesimal thickening of order $\leq n$ of $R$ is a pair $(A,I)$, where $A$ is a $\Lambda$-algebra, $I \subset A$ is an ideal, together with an isomorphism $A/I \to R$ of $\Lambda$-algebras, such that $I^{n+1} = 0$. It is natural to ask whether the category of such thickenings has an initial object. The answer is no in general, so we consider the category of triples $(A,I,\sigma)$, where $(A,I)$ is a $\Lambda$-thickening of $R$ and $\sigma : R \to A$ is a multiplicative section of $A \to R$. The category of such triples has an initial object which can be constructed explicitly.

Start with the monoid ring $\Lambda[R]$ consisting of formal sums $\sum_{x \in R} \lambda_x x$, with all but finitely many $\lambda_x = 0$. There is a ring homomorphism (called the augmentation map) $\varepsilon : \Lambda[R^\times] \to R$, sending $\sum \lambda_x x$ to $\sum \lambda_x x$. Put $U_{n,\Lambda}(R) = \Lambda[R^\times]/(\ker \varepsilon)^{n+1}$. Clearly $(U_{n,\Lambda}(R), \ker \varepsilon)$ is a $\Lambda$-infinitesimal thickening of $R$ of order $\leq n$. There is an obvious multiplicative section $\sigma : R \to U_{n,\Lambda}(R)$ given by $r \mapsto [r]$. It is easy to check that $(U_{n,\Lambda}(R), \ker \varepsilon, \sigma)$ is the universal “$\Lambda$-infinitesimal thickening of order $\leq n$ with section” of $R$.

If $R$ is a perfect $F_p$-algebra, then $U_{n,\mathbb{Z}}(R) = U_{n,\mathbb{Z}_p}(R)$ is $W_n(R)$, the ring of length-$n$ Witt vectors of $R$, so $W(R) = \varprojlim_n U_{n,\mathbb{Z}}(R)$. If $R$ is a perfect $F_p$-algebra, then given any $\Lambda$-infinitesimal thickening $(A,I)$ of $R$, we can form a multiplicative section $\sigma : R \to A$ by $x \mapsto (x^{p^{-m}})^{p^m}$, for $x^{p^{-m}}$ a lifting of $x$ to $A$, and $m \gg 0$.

5.2 Tilting revisited

Let’s start by defining the class of rings we’ll be working with.

**Definition 5.2.1.** A Banach ring $A$ is a topological ring containing a pseudo-uniformizer (topologically nilpotent invertible element) $\pi$, and an open subring $A_0 \ni \pi$, such that $A = A_0[\frac{1}{\pi}]$, and such that $A_0 \to \varprojlim A_0/\pi^n$ is an isomorphism of topological rings.

Note that a Banach ring is simply a complete Tate algebra (in the sense of Huber).

Recall that if $A$ is a Banach ring, then a set $S \subset A$ is called bounded if for all neighborhoods $U$ of 0, there exists $n \geq 1$ so that $\pi^n S \subset U$. An element $a \in A$ is power-bounded if the set $\{a^n : n \geq 0\}$ is bounded, and we write $A^\circ \subset A$ for the subring of power-bounded elements.

**Definition 5.2.2.** A spectral ring is a Banach ring $A$ such that $A^\circ$ is bounded.

If $A$ is a Banach ring, then $A$ is spectral if and only if there exists a power-multiplicative norm inducing the topology. Recall that a **power-multiplicative norm** on $A$ is a map $| \cdot | : A \to \mathbb{R}_{\geq 0}$ such that $|ab| \leq |a| \cdot |b|$, $|1| = 1$, and $|a^n| = |a|^n$ for $n \geq 0$. If $A$ is complete with respect to $| \cdot |$, one has $A^\circ = \{a \in A : |a| \leq 1\}$. If $A$ is a Banach ring, we can construct a
power-multiplicative norm directly. Choose $\pi \in A$ as in Definition 5.2.1. Also choose $\rho \in \mathbb{R}$ with $0 < \rho < 1$. For $a \in A$, put
\[ v_\pi(a) = \sup \left\{ \frac{r}{s} : r \in \mathbb{Z}, s > 0 \text{ and } \frac{a^s}{\pi^r} \in A^\circ \right\}. \]

One has $v_\pi(a) \in \mathbb{R}_{\geq 0}$, so it makes sense to define $|a| = \rho^{v_\pi(a)}$.

**Definition 5.2.3.** A perfectoid ring is a spectral ring $A$, such that there exists a power-bounded unit $\pi \in A^\circ$ such that $p \in \pi^p A^\circ$ and the Frobenius map $A^\circ / \pi^p A^\circ$ is surjective.

So a perfectoid field is a perfectoid ring which is a field, and which admits a multiplicative norm.

If $A$ is a spectral ring of positive characteristic, then $A$ is perfectoid if and only if it is perfect. Moreover, if $\pi \in A$ is any pseudo-uniformizer, then $A$ is naturally an algebra over the perfectoid field $\mathcal{F}(\mathbb{Q}|_{1/p^\infty}) = \mathcal{F}(\pi)(\pi^{1/p^\infty})$.

If the characteristic of $A$ is not $p$, then $A$ is a Banach $p$-adic algebra, so $\mathbb{Q}_p \hookrightarrow A$, but there is no canonical perfectoid subfield of $A$.

Let $A$ be a perfectoid ring of characteristic zero. One can define a new ring $A^b$, called the *tilt* of $A$, in two different but equivalent ways. First, choose a pseudo-uniformizer $\pi \in A^\circ$ with $p \in \pi A^\circ$, and put
\[ A^b = \lim_{\Phi} A^\circ / \pi \]
where $\Phi : x \mapsto x^p$ is Frobenius. Define $A^b = A^\circ[\frac{1}{\pi}]$. Alternatively, we could have defined $A^b$ as a set to be
\[ A^b = \lim_{x \mapsto x^p} A = \left\{ (x^{(n)})_{n \geq 0} : (x^{(n+1)})^p = x^{(n)} \right\}. \]

This has an obvious multiplication operation, but the addition is not obvious. We define it directly by
\[ (x + y)^{(n)} = \lim_{m \to \infty} (x^{(n+m)} + y^{(n+m)})^{p^m}. \]

Define $(-)^{b_\pi} : A^b \to A$ given by $(x^{(n)})_{n \geq 0} \mapsto x^{(0)}$.

If $A$ is a perfectoid ring of characteristic zero, put $R = A^b$; this is a perfectoid ring of characteristic $p$. There is a natural surjection $\theta_A : W(R^\circ) \twoheadrightarrow A^\circ$, defined by
\[ \theta_A \left( \sum_{i=0}^\infty p^i [a_i] \right) = \sum_{i=0}^\infty p^i a_i^{(0)} = \sum_{i=0}^\infty p^i a_i^{(0)} \]
where each $a_i = (a_i^{(n)})_{n \geq 0}$. The kernel of $\theta_A$ is a primitive ideal of degree one. In other words, it can be generated by an element of the form $[\pi] + p\eta$, where $\pi$ is a pseudo-uniformizer of $R$, and $\eta \in W(R^\circ)^\times$.

The operation $A \mapsto (A^b, \ker \theta_A)$ is a functor from the category of perfectoid rings of characteristic zero to the category of “perfectoid pairs.” Here a perfectoid pair is a pair $(R, I)$, where $R$ is a perfectoid ring of characteristic $p > 0$, and $I \subset W(R^\circ)$ is a primitive ideal of degree one.

**Theorem 5.2.4.** This functor is an equivalence of categories.

**Proof.** We construct an inverse. Given a perfectoid pair $(R, I)$, define $A = (R, I)^{b_\pi}$ by $A^\circ = W(R^\circ)/I$ and $A = A^\circ[\frac{1}{p}]$. Checking that $(-)^{b_\pi}$ is an inverse to $A \mapsto (A^b, \ker \theta_A)$ is a straightforward exercise. Alternatively, see Kedlaya’s note [Ked13].
5.3 Tilting and analytic extensions

We know that for any perfectoid field $K$ of characteristic zero, the tilt $K^\flat$ is a perfectoid field of characteristic $p$. On the other hand, suppose we start out with a perfectoid field $F$ of characteristic $p$. Then to any primitive ideal $I \subset W(F^\circ)$, the above construction yields a perfectoid field $(F, I)^\flat$ of characteristic zero, whose tilt is $F$. Thus every perfectoid field of characteristic $p$ is the tilt of a (explicitly constructed) perfectoid field of characteristic zero.

Suppose $K_0$ is a finite extension of $\mathbb{Q}_p$. Let $L$ be an “analytically profinite extension of $K_0$.” In other words, $L$ could be an algebraic extension of $K_0$ which is infinitely ramified and such that the Galois group of the Galois closure of $L/K_0$ is a $p$-adic Lie group. Clearly any analytically profinite extension of $K_0$ gives a $p$-adic representation of $G_{K_0}$.

Let $K$ be the completion of $L$. To $K$ we associate a “norm field,” which is isomorphic to $k_L(\langle t \rangle)$, where $k_L$ is the residue field of $L$. The “Fontaine-Wintenberger theorem” is that $L$ and $k_L$ have the same étale sites. Moreover, $K$ is perfectoid and $K^\flat$ is the perfectoid completion of $k_L(\langle t \rangle)$.

If we took $K = \mathbb{C}_p$, then we get an algebraically closed perfectoid field $F$ of characteristic $p$. It is unknown if all the untiltings of this field are isomorphic.

Let $K$ be a fixed perfectoid field of characteristic zero. Then $K^\flat$ is a perfectoid field of characteristic $p$. We know that the kernel of $\theta_K : W(K^\flat) \to K^\circ$ is principal; let $\xi$ be any generator. If $A$ is any perfectoid $K$-algebra, then $(A^\flat, (\xi))$ is a perfectoid pair.

5.4 Period rings

Let $k$ be a perfect field of characteristic $p$, and fix an ultrametric field $E$ with residue field $k$. For any perfect $k$-algebra $R$ of characteristic $p$, we will construct a “bounded period ring” $B_E^b(R)$. Start by defining $W_{E^\circ}(R)$ to be the completed tensor product

$$E^\circ \otimes_{W(k)} W(R) = \lim_{\longleftarrow} (E^\circ / \pi^n) \otimes_{W(k)} W(R),$$

where $\pi \in E^\circ$ is any pseudo-uniformizer.

The surjection $W_{E^\circ}(R) \to R$ has a multiplicative section $[-] : R \to W_{E^\circ}(R)$, given by $x \mapsto 1 \otimes [x]$. If $E$ has characteristic $p$, this section is additive and we can view $R$ as a subring of $W_{E^\circ}(R) = E^\circ \otimes_k R$.

Suppose $R$ is a perfectoid $k$-algebra. (Since $k$ is not perfectoid, we simply assume $k \subset R^\circ$.) Let $\varpi$ be a pseudo-uniformizer of $R$, and define

$$B_E^b(R) = W_{E^\circ}(R^\circ) \left[ \begin{array}{c} 1 - \frac{1}{\pi}, 1 - \frac{1}{\varpi} \end{array} \right].$$

If $E$ is discretely valued, we can choose $\pi$ to be a uniformizer of $E$, and

$$B_E^b(R) = \left\{ \sum_{i, \varpi^{-n} < \infty} [a_i] \pi^i : a_i \in R \text{ bounded} \right\}.$$

In general, given $(a_n)_{n \geq 0}$, with $\{a_n\} \subset R$ bounded, and $(\nu_n)_{n \geq 0}$, with $\nu_n \in E$ converging to zero, the infinite sum $\sum_{n=0}^{\infty} [a_n] \nu_n \in B_E^b(R)$, but this “decimal expansion” is not unique.

Choose an absolute value on $E$, and a power-multiplicative norm on $R$. Up to equivalence, these are unique.

**Proposition 5.4.1.** For all $\rho \in \mathbb{R}$, $0 < \rho < 1$, there exists a unique power-multiplicative norm $| \cdot |_\rho$ on $B_E^b(R)$ such that $|a|_\rho = |a|$ for all $a \in R$, and such that $|\pi| = \rho$. 

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Choose a non-empty closed interval $I = [\rho_1, \rho_2]$, with $0 < \rho_1 \leq \rho_2 < 1$. Define

$$|\alpha|_I = \sup\{|\alpha|_{\rho_1}, |\alpha|_{\rho_2}\} = \sup_{\rho_1 \leq \rho \leq \rho_2} |\alpha|_{\rho}.$$  

Let $B_{E/I}(R)$ be the completion of $B_{E}^+(R)$ with respect to the norm $|\cdot|_I$. This is a spectral $E$-algebra. If $E$ is a perfectoid field, then $B_{E/I}(R)$ is a perfectoid $E$-algebra. The tilt of $B_{E/I}(R)$ is $B_{E^+/I}(R)$.

Now we can define some period rings. For any perfectoid $\mathbb{Q}_p$-algebra $R$, we can define $B_{\text{cris}}^{+}(A)$ and $B_{\text{dR}}^{+}(A)$. The “morally correct” definition of $B_{\text{dR}}^{+}$ is

$$B_{\text{dR}}^{+}(A) = \lim_{\leftarrow n} B_n(A).$$  

Here the inverse limit is taken in the category of Banach $\mathbb{Q}_p$-algebras, and $B_n(A)$ is the universal divided-power thickening of $A$ of order $\leq n + 1$. For a more explicit definition, start with the homomorphism $\theta : W(A^{\varphi})[\frac{1}{p}] \to A$, whose kernel is a principal ideal $(\xi)$. It turns out that $B_n(A) = W(A^{\varphi})[\frac{1}{p}]/\xi^n$, so

$$B_{\text{dR}}^{+}(A) = \lim_{\leftarrow \xi} W(A^{\varphi}) \left[\frac{1}{p}\right]/(\ker \theta)^n.$$  

This is a discrete valuation ring with uniformizer $\xi$, so we can define $B_{\text{dR}}^{+}(A)$ to be the field of fractions of $B_{\text{dR}}^{+}(A)$, i.e. $B_{\text{dR}}^{+}(A) = B_{\text{dR}}^{+}(A)[\frac{1}{\xi}]$.

To define $A_{\text{cris}}^{+}(A)$, start by defining $A_{\text{cris}}^{\varphi}(A)$ to be the sub-$W(A^{\varphi})$-algebra of $W(A^{\varphi})$ generated by $\{\xi^n/n : n \geq 1\}$. Let $A_{\text{cris}}(A)$ be the $p$-adic completion of $A_{\text{cris}}^{\varphi}(A)$, and define $B_{\text{cris}}^{+}(A) = A_{\text{cris}}(A)[\frac{1}{p}]$.  

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6 Adic spaces 3

6.1 Étale topology for perfectoid spaces

Note that perfectoid rings (i.e. spectral rings, in Fontaine’s vocabulary) are always reduced. If \( x \in R \) is nilpotent, then any \( \varpi^{-n}x \in R^0 \), so \( K \cdot x \subset R^0 \), but this is impossible if \( R^0 \) is bounded. So we can’t use an infinitesimal lifting criterion to define “étaleness.” However, we have the following proposition.

**Proposition 6.1.1.** For adic spaces of finite type over \( K \), open embeddings and finite étale covers generate the étale site.

Note that this is not true for schemes. One defines finite étale covers using Bhargav’s definition for global sections.

**Definition 6.1.2.** A map \( f : Y \to X \) of perfectoid spaces is étale if it is locally of the form \( Y \hookrightarrow Z \to W \to X \), where \( Y \hookrightarrow Z \) is an open embedding, \( Z \to W \) is finite étale, and \( W \to X \) is open.

One can check that this notion of étaleness satisfies the expected properties (e.g. closure under composition). Thus we can define the étale site \( X_{\text{ét}} \). Moreover, if a map becomes étale after a change of the ground field, it was étale to begin with.

**Proposition 6.1.3.** If \( X \) is a perfectoid space with tile \( X^\flat \), then

1. \( X_{\text{ét}} \simeq X^\flat_{\text{ét}} \)

2. Let \( X \) be affinoid perfectoid; \( X = \operatorname{Spa}(R, R^+) \). Then

\[
H^i(X_{\text{ét}}, \mathcal{O}^+_X) = \begin{cases} R^+ & \text{if } i = 0 \\ \text{almost zero} & \text{if } i > 0 \end{cases}
\]

Part 1 follows from the fact that \( |X| \simeq |X^\flat| \), combined with the almost purity theorem for finite étale covers.

If \( X = \operatorname{Spa}(R, R^0) \) is affinoid of finite type over \( K \), then \( H^i(X_{\text{ét}}, \mathcal{O}_X) \) is \( R \) for \( i = 0 \), and is \( 0 \) for \( i > 0 \). On the other hand, \( H^i(X_{\text{ét}}, \mathcal{O}_X^+) \) can contain lots of torsion, e.g. it could contain \( K/\mathcal{O}_K \). This torsion gets killed on perfectoid covers.

In Niziol’s talks, we will see the following theorem. Let \( K \) be algebraically closed of characteristic 0, and let \( X \) be a proper smooth variety over \( K \). We can look at \( R\Gamma(X_{\text{ét}}, \mathcal{O}_X^+) \).

One has \( R\Gamma(X_{\text{ét}}, \mathcal{O}^+_X)(\mathbb{L}) = R\Gamma(X_{\text{ét}}, \mathcal{O}_X) \) (usual coherent cohomology). On the other hand, \( R\Gamma(X_{\text{ét}}, \mathcal{O}_X^+) \otimes_{\mathcal{O}_K} \mathcal{O}_K/p^n \) is almost isomorphic to \( R\Gamma(X_{\text{ét}}, \mathbb{Z}/p^n) \otimes_{\mathbb{Z}/p^n} \mathcal{O}_K/p^n \). This gives us a connection between cohomology \( \mathcal{O}_X^+ \) on a perfectoid space, and more familiar étale cohomology.

It is natural to ask whether there is a “good” category containing both rigid-analytic varieties and perfectoid spaces. In some sense, the answer is “yes” because the category of adic spaces accomplishes this. On the other hand, this is not a satisfactory answer because the category of adic spaces is not as nice as one might hope. The problem is that it is very hard to determine whether \( \mathcal{O}_{X} \) (or quasi-coherent modules) are sheaves.

Peter Scholze (Feb. 18)
Example 6.1.4 (Rost, Buzzard). Consider $R = \mathbb{Q}_p(pT, pT^{-1})$; the algebra of functions on the strip $p^{-1} \leq |T| \leq p$. Let $M$ be the Banach $R$-module with Banach $\mathbb{Q}_p$-basis $\{p^{-n} \cdot T^{-n}, p^{-n}T^n : n \geq 0\}$. In other words, $M$ is the $p$-adic completion of the submodule of $\mathbb{Q}_p(T^\pm 1)$ generated by these elements. Let $R_1 = \mathbb{Q}_p(pT, T^{-1}), R_2 = \mathbb{Q}_p(T, pT^{-1})$.

Proposition 6.1.5. The element $1 \in M$ dies in $M \hat{\otimes}_R R_i$ for $i = 1, 2$. In fact, $M \hat{\otimes}_R R_i = 0$ for each $i$.

Proof. We can assume $i = 1$. Write $p^{-n} = (p^{-n}T^n) \cdot T^{-n} \in M \cdot R_1$. One has $|p^{-n}T^n| \leq 1$ and $|T^{-n}| \leq 1$. By the definition of the norm on $M \hat{\otimes}_R R_i$, one gets $|p^{-n}| \leq 1$ for all $n$, i.e. $|1| \leq p^{-n}$ for all $n$, whence $|1| = 0$, so $1$ dies.

Theorem 6.1.6. $R \oplus M$ (M a square-zero ideal) violates the sheaf property.

Proof. In fact, there is an element which is globally non-zero, but is locally zero. Note that $R \oplus M$ is not spectral in the sense of Fontaine, because it has nilpotents.

Proposition 6.1.7. If $R$ is spectral, then for any cover $X = \text{Spa}(R, R^+)$ with the $U_i$ rational, one has $R \twoheadrightarrow \prod \mathcal{O}_X(U_i)$. In fact, $R \twoheadrightarrow \prod_{x \in X} \hat{k}(x)$.

Proof. This follows from a theorem of Berkovich to the effect that the pullback of the supremum norm on $\prod \hat{k}(x)$ makes $R^\oplus$ the $|\cdot| \leq 1$ subring.

6.2 Open problems

A natural question is: are there counterexamples to the sheaf property for spectral rings? Also, is there spectral $R$ with $U \subset \text{Spa}(R, R^+)$ such that $\mathcal{O}_X(U)$ is not spectral? Finally, could these two phenomena occur simultaneously?

The following question is due to Rapoport: is “perectoid” a local property? Let $K$ be a perfectoid field. Let $X$ be a perfectoid space over $K$. Assume $X = \text{Spa}(R, R^+)$ is affinoid. Is $R$ perfectoid? We know that there is a covering of $X$ by rational subsets which are perfectoid, but it is not clear that this implies the “perfectoidness” of $X$. Do we have to distinguish between “affinoid” and “perfectoid affinoid” subsets of a perfectoid space?

One can consider inverse limits. Let $(X_i)_{i \geq 0}$ be a tower of reduced adic spaces, all of finite type over a field $K$. Moreover, we assume that the transition maps are finite. Let $X$ be a perfectoid space, and $\{f_i : X \to X_i\}_{i \geq 0}$ a compatible system of maps.

Definition 6.2.1. 1. Say that $X$ is a naive inverse limit if all $X_i = \text{Spa}(R_i, R_i^+)$ and $X = \text{Spa}(R, R^+)$, everything is affinoid perfectoid, and $R^+$ is the $\pi$-adic completion of $\lim R_i^+$.

2. Say $X \sim \varprojlim X_i$ if this is satisfied locally.

The category of affinoid rings does not admit filtered direct limits. If $\{(R_i, R_i^+)\}_i$ is a direct system, what topology should we put on $\varprojlim R_i^+$? With the direct limit topology, $\varprojlim R_i^+$ is not affinoid, and it is not clear what other topology to impose. On the other hand, if the $R_i$ are spectral, we can let $R^+$ be the $\omega$-adic completion of $\varprojlim R_i^+$ and $R = R^+[\frac{1}{\omega}]$.

Proposition 6.2.2. If $X \sim \varprojlim X_i$, then $X = \varprojlim X_i$ in the category of locally spectral adic spaces.
Another question: does it make sense to develop the theory of spectral adic spaces? Spectral adic spaces are reduced, so they “don’t see” tangent spaces, at least not via \( K[[\varepsilon]]/\varepsilon^2 \) valuated points. For example, the isomorphism of Lubin-Tate and Drinfel’d towers, we have

\[
\lim_{\to} M_i \sim M_\infty \simeq \lim_{\to} N_i
\]

But is \( \Omega^1_{M_0} |_{M_\infty} \simeq \Omega^1_{N_0} |_{N_\infty} \)? This is not known.

### 6.3 Generic fibers of formal schemes

This section is preparation for Weinstein’s talk. Let \( R \) be complete for the \( I \)-adic topology for some finitely generated ideal \( I \subset R \). One can consider \( \text{Spa}(R, R) \), which comes equipped with a structure-presheaf. Assume \( R \) is a \( \mathbb{Z}_p \)-algebra, and \( p \in I \). Then \( \text{Spa}(R, R) \) lies over \( \text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p) = \{s, \eta\} \), where \( s \) is the special point and \( \eta \) is the generic point. One has \( \eta = \text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \).

**Definition 6.3.1.** The generic fiber of \( \text{Spf} R \) is \( \text{Spa}(R, R)_\eta = \text{Spa}(R, R) \times_{\text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)} \text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \).

The generic fiber is an open subset of \( \text{Spa}(R, R) \).

In any reasonable setup, \( \text{Spf} R \mapsto \text{Spa}(R, R) \) is a fully faithful functor \( \{\text{formal schemes}\} \hookrightarrow \{\text{adic spaces}\} \).

**Example 6.3.2.** Suppose \( R = \mathbb{Z}_p\langle T \rangle \). Then

\[
\text{Spa}(R, R)_\eta = \text{Spa}(R[p^{-1}], R) = \text{Spa}(\mathbb{Q}_p\langle T \rangle, \mathbb{Z}_p(T))
\]

the closed unit disk.

**Example 6.3.3.** If \( R = \mathbb{Z}_p[[T]] \) with \( I = (p, T) \), then \( \text{Spa}(R, R)_\eta \) is the open unit disk, hence is not affinoid.

**Example 6.3.4.** Let \( R = \mathcal{O}_K \left[ T^{1/p^n} \right] \), the \( (p, T) \)-adic completion of \( \mathcal{O}_K[T^{1/p^n}] \), where \( K \) is some perfectoid field. Then \( \text{Spa}(R, R)_\eta \) is the “perfectoid open unit ball.”

**Proposition 6.3.5.** Fix \( f_1, \ldots, f_k \in I \) such that \( I = (p, f_1, \ldots, f_k) \). Then

\[
\text{Spa}(R, R)_\eta = \bigcup_{n \geq 1} \text{Spa}(R(\frac{f_1^n}{p}, \ldots, \frac{f_k^n}{p})(\frac{1}{p}))
\]

where we assume \( |f_i| \leq |p|^{1/n} \).

**Proposition 6.3.6.** Assume \( R \) is a flat commutative \( \mathcal{O}_K \)-algebra, where \( K \) is a perfectoid field. Moreover, assume \( \Phi : R/p^{1/p} \to R/p \) is an isomorphism. Then \( \text{Spa}(R, R)_\eta \) is a perfectoid space over \( K \).
7 The weight-monodromy conjecture

The goal is to understand Scholze’s proof of the weight-monodromy conjecture for hypersurfaces in projective space. We will do this by comparing étale topoi in characteristic zero and characteristic \( p \).

7.1 Tilting

Recall the tilting equivalence. Let \( K \) be a perfectoid field (i.e. a complete non-archimedean field with non-discrete valuation such that Frobenius \( K^\circ /\pi \to K^\circ /\pi \) is surjective, where \( \pi \in K^\circ \) has \( |p| \leq |\pi| < 1 \)). The tilt of \( K \) is \( K^\flat \), which as a set is

\[
K^\flat = \lim_{x \to x^p} K.
\]

There is a section map \((-)^!: K^\flat \to K\) that is multiplicative and continuous, but not additive or surjective. We say that \( \text{Spa}(K,K) \) tilts \( K \) to \( \text{Spa}(K^\flat,K^\flat) \). A baby case of the almost purity theorem tells us that there is an equivalence of categories \( K_{\text{ét}} \simeq K^\flat_{\text{ét}} \). This leads to an isomorphism of Galois groups \( \text{Gal}(\bar{K}/K) \simeq \text{Gal}(K^\flat/K^\flat) \).

This can be generalized to perfectoid spaces over \( K \). Recall that a perfectoid space over \( K \) is an adic space \( X \) over \( K \) that is locally of the form \( \text{Spa}(R,R^+) \), where \( R \) is a perfectoid \( K \)-algebra and \( R^+ \subseteq R^\circ \) is an open integrally closed subring. The space \( X \) tilts (locally) to \( \text{Spa}(R^\flat,R^\flat^+) \), yielding a functor \( X \to X^\flat \) from perfectoid spaces over \( K \) to perfectoid spaces over \( K^\flat \).

Recall that we have a map \( R^\flat = \lim_{x \to x^p} R \to R \) given by \( (x_i)_{i \geq 1} \mapsto x_0 \). This map is continuous, multiplicative, but not surjective. However, it induces an isomorphism \( R^+/\pi \simeq R^\flat^+/\pi^\flat\). This lets define a homeomorphism \( |X| \to |X^\flat|, x \mapsto x^\flat \), where for \( f \in R^\flat \), we have \( |f(x^\flat)| = |f(x)| \). This homeomorphism preserves rational subsets. To prove this, one uses an approximation lemma: given \( f \in R^+ \), there exists \( g \in R^\flat^+ \) such that \( |f + |g| \) except when both are “very small” (i.e. except when \( |f| < \varepsilon \), in which case we only require \( |g^\flat| < \varepsilon \).

The structure sheaves \( (\mathcal{O}_X, \mathcal{O}_X^\flat) \) of \( X \) tilt to sheaves \( (\mathcal{O}_{X^\flat}, \mathcal{O}_{X^\flat}^\flat) \) on \( X^\flat \). Recall that an étale morphism of perfectoid spaces is a composite of a finite étale morphism with an open immersion. Using the almost purity theorem, we get an isomorphism \( X_{\text{ét}} \simeq X^\flat_{\text{ét}} \). Recall that a key point here is that locally, \( \mathcal{O}_{X,x}^\flat[\frac{1}{x}] \simeq \mathcal{O}_{X^\flat,x}^\flat[\frac{1}{x}] \), and both are perfectoid fields. This reduces the proof of the almost purity theorem to the case for perfectoid fields. This isomorphism on stalks actually holds more generally for analytic adic spaces. The sheaf property is crucial here.

7.2 Comparing perfectoid spaces with noetherian spaces

To apply any of this to the weight-monodromy conjecture, we need to compare étale sites and underlying spaces of perfectoid spaces with the étale sites and underlying spaces of locally noetherian adic spaces. For example, let \((\mathbb{P}^n_K)^{\text{ad}}\) be the adic projective space glued out of \( \text{Spa}(K(T_1, \ldots, T_n), K^\circ(T_1, \ldots, T_n)) \) in the usual way. Also define \((\mathbb{P}^n_K)^{\text{perf}}\) by glueing copies of

\[
\text{Spa}(K(T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}), K^\circ(T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}))
\]
in the usual way.

**Definition 7.2.1.** Let $X$ be a perfectoid space over $K$, and \{ $X_i$ \} a filtered inverse system of noetherian adic spaces over $K$, together with a compatible system of maps $\varphi_i : X \to X_i$. We say that $X \sim \lim \limits_{\leftarrow} X_i$ if

1. The induced map $|X| \to \lim |X_i|$ is a homeomorphism.
2. For all $x \in X$ inducing $x_i \in X_i$, the map $\lim \limits_{\leftarrow} k(x_i) \to k(x)$ has dense image.

Note that if $Y \to X_i$ is an étale morphism of adic spaces, then $Y \times_{X_i} X \sim \lim \limits_{\leftarrow} j \geq i, Y \times_{X_i} X_j$.

Consider $\mathbb{P}^n_{K, \text{perf}}$ as perfectoid space over $K$.

**Theorem 7.2.2.** We have $\mathbb{P}^n_{K, \text{perf}} \simeq \mathbb{P}^n_{\overline{K}}$.

2. $\mathbb{P}^n_{K} \simeq \lim \limits_{\leftarrow} \mathbb{P}^n_{K, \text{ad}}$, where $\Phi(x_0 : \cdots : x_n) = (x_0^p : \cdots : x_n^p)$.
3. There are homeomorphisms of topological spaces $|\mathbb{P}^n_{K^K}| \simeq |\mathbb{P}^n_{\text{perf}}| \simeq |\mathbb{P}^n_{K}| \simeq \lim \limits_{\leftarrow} \mathbb{P}^n_{K, \text{ad}}$.
4. There is an equivalence of étale topoi $\mathbb{P}^n_{K, \text{ad}, \text{ét}} \simeq \lim \limits_{\leftarrow} \mathbb{P}^n_{K, \text{ad}, \text{ét}}$.
5. Denote by $\pi$ the map $|\mathbb{P}^n_{K^K}| \to |\mathbb{P}^n_{K}|$. If $U \subset |\mathbb{P}^n_{K}|$ is open, put $V = \pi^{-1}(U) \subset |\mathbb{P}^n_{K^K}|$. There is a commutative diagram

\[
\begin{array}{ccc}
V_{\text{ét}} \rightarrow & \mathbb{P}^n_{\text{ad}, \text{ét}} \\
\downarrow & \\
U_{\text{ét}} \rightarrow & \mathbb{P}^n_{K, \text{ét}}
\end{array}
\]

**Proof.** 1. It suffices to check this on affinoid pieces, where this comes down to the fact that

\[
(K^\circ/\pi)(T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}) = (K^\circ/\pi^3)(T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}).
\]

3. This follows from part 2 and the tilting equivalence. \hfill $\square$

Note that part 3 tells us that $|\mathbb{P}^n_{K^K}| \simeq \lim \limits_{\leftarrow} \mathbb{P}^n_{K}$.

There is a map $\lim \limits_{\leftarrow} \mathbb{P}^n_{K, \text{ad}} \to |\mathbb{P}^n_{K^K}|$ that on coordinates is $(x_0 : \cdots : x_n) \mapsto (x_0^p : \cdots : x_n^p)$.

**Corollary 7.2.3.** There are natural isomorphisms

\[
H^i(\mathbb{P}^n_{K, \text{ad}, \text{ét}}, \mathbb{Z}/\ell^m) \simeq H^i(\mathbb{P}^n_{\text{ad}, \text{ét}}, \mathbb{Z}/\ell^m)
\]

In proving the corollary, one can assume that $K$ (and hence $K^\circ$) are separably closed.

### 7.3 Proving the weight-monodromy conjecture

Let $k$ be a local field, $X$ a proper smooth variety over $k$. Then the groups $V = H^i(X_k, \mathbb{Q}_\ell)$ admit a continuous action of the group $\text{Gal}(\overline{k}/k)$. Assume $\ell \neq p$. Then this action is characterized by the action a (lift of) Frobenius. Let $\pi \in k$ be a uniformizer and $\mathbb{F}_q = k^\circ/\pi$. 

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By the Grothendieck ℓ-adic monodromy theorem, after a finite base-change, everything is defined in terms of a nilpotent operator \( N \) (which encodes the action of the tame inertia).

From basic linear algebra, we know that we can write \( V = \bigoplus_{j=0}^{2r} V_j \), where on each \( V_j \) the eigenvalues of Frobenius are Weil numbers of weight \( j \) (i.e. their absolute value is \( q^{j/2} \) for each complex embedding \( \bar{Q}_\ell \to \mathbb{C} \)). If \( X \) has good reduction, we know that only one weight occurs, so \( N = 0 \). This is because in general, \( N : V_j \to V_{j-2} \).

**Conjecture 7.3.1** (Deligne). If \( V \) is as above, then \( N^j : V_{i+j} \to V_{i-j} \) is an isomorphism for all \( 0 \leq j \leq i \).

Equivalently, we can define the monodromy filtration on \( V \) which is uniquely characterized by the conclusion of the conjecture, and the conjecture claims that the monodromy filtration and weight filtration are the same.

The weight-monodromy conjecture is known in the equal-characteristic case.

**Theorem 7.3.2** (Deligne). Let \( C \) be a curve over \( \mathbb{F}_q \), \( x \in C(\mathbb{F}_q) \) such that \( k \) is the local field of \( C \) at \( x \). If \( X \to C \setminus \{x\} \) is smooth, then \( X_k = X \times_{C \setminus \{x\}} \text{Spec}(k) \) satisfies the weight-monodromy conjecture.

Note that the weight-monodromy conjecture only concerns the monodromy operator and the Frobenius operator. If we let \( K = k(1/p^{\infty}) \), then these remain the same, so it suffices to work with varieties over \( K \).

Let \( Y \subset \mathbb{P}_K^n \) be a smooth hypersurface. We can pass to the adic world to obtain \( Y^{\text{ad}} \subset \mathbb{P}_K^n \). There is a comparison theorem between the étale cohomology of \( Y \), and the étale cohomology of an open adic neighborhood \( \tilde{Y} \) of \( Y^{\text{ad}} \). Over \( \mathbb{C}_p \), we have a diagram

\[
\begin{array}{ccc}
\mathbb{P}_\mathbb{C}_p^n & \longrightarrow & \tilde{Y}_{\mathbb{C}_p} \\
\uparrow & & \uparrow \\
\pi^{-1}(\tilde{Y}) & \longrightarrow & Y_{\mathbb{C}_p}
\end{array}
\]

**Lemma 7.3.3.** Suppose \( Y \) is cut out by a homogeneous polynomial \( f \) of degree \( d \). Then there exists a homogeneous polynomial \( g \in \mathbb{C}_p[\{T_1^{1/p^{\infty}}, \ldots, T_n^{1/p^{\infty}}\}] \) such that \( |f(x)| \leq \epsilon \) if and only if \( |g^\ell(x)| \leq \epsilon \) for all \( x \in \mathbb{P}_\mathbb{C}_p^n \).

It turns out that \( \pi^{-1}(\tilde{Y}) \) will contain some \( Z \) algebraic over \( \mathbb{C}_p \). Huber gives comparison isomorphisms \( H^i(Y, \mathbb{Z}/\ell^m) \simeq H^i(\tilde{Y}, \mathbb{Z}/\ell^m) \) for \( \tilde{Y} \) sufficiently small. In this setting, we get an injection

\[
H^i(Y_{\mathbb{C}_p,\text{ét}}, \mathbb{Q}_\ell) \longrightarrow H^i(Z_{\mathbb{C}_p,\text{ét}}, \mathbb{Q}_\ell)
\]

that is \( \text{Gal}(\overline{K}/K) \simeq \text{Gal}(\overline{K}^\text{ad}/K^\text{ad}) \)-equivariant and compatible with cup product.

One can choose \( Z \) so that the representation on the right satisfies the weight-monodromy conjecture, so it suffices to show that the space on the left is a direct summand. We use this using Poincaré duality and the following lemma.

**Lemma 7.3.4.** Let \( f \in K^\text{ad}[\{T_0^{1/p^{\infty}}, \ldots, T_n^{1/p^{\infty}}\}] \) be a homogeneous polynomial of degree \( d \). Then for all \( \epsilon, c > 0 \), there exists \( g \in K^\text{ad}[\{T_0^{1/p^{\infty}}, \ldots, T_n^{1/p^{\infty}}\}] \) such that \( |f(x) - g^\ell(x)| \leq |\pi|^{1-\epsilon} \cdot \max(|\pi|^c, |f(x)|) \).

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Proof. Use induction on $c$. Let $R^\circ = K\langle T_0^{1/p^\infty}, \ldots, T_\infty^{1/p^\infty} \rangle$. We know that $R^\circ / \pi \simeq R^\circ / \pi^c$. Almost mathematics tells us that if $|f(x)| \leq |\pi|^c$, then $f - g \in |\pi|^{1-c} T_1(U)$.

To conclude, we get a map $H^i(Y, \mathbb{Q}_\ell) \to H^i(Z, \mathbb{Q}_\ell)$. To show this is a direct summand, it is enough to look at $i = 2 \dim Y$, where the map is either an isomorphism or zero. If the map is zero, we get that $H^i(\mathbb{P}^n_{\mathbb{C}_p}, \mathbb{Q}_\ell) \to H^i(Z_{\mathbb{C}_p}, \mathbb{Q}_\ell)$ is zero, but this cannot happen (because of Chern classes). Thus the top-degree map is an isomorphism, so we use compatibility with cup-products and Poincaré duality to get the result.
8 Lubin-Tate spaces 1

Let’s move from studying perfectoid spaces “in the abstract” to examining some perfectoid spaces that arise “in nature.”

Consider $X(Np^m)$ as a rigid space. The supersingular locus of $X(Np^m)$ is a disjoint union $\coprod_{m \geq 0} \mathcal{M}_m$, where each $\mathcal{M}_m$ is the Lubin-Tate space of level $m$. Letting $m \to \infty$, everything in sight becomes a perfectoid space.

8.1 The Lubin-Tate deformation problem

Let $k$ be an algebraically closed field of characteristic $p$. Let $H_0$ be a one-dimensional formal group over $k$. Assume $H_0$ has finite height $n$. We can write $X + H_0 = X + Y + \cdots$, and the condition on height is that $[p]_{H_0}(T) = f(T^{p^n})$, where $f(T) = cT + \cdots$, where $c \in k^\times$. Better, we can consider $H_0$ as $\text{Spf}(k[[T]])$, and $H_0$ is a group object in the category of formal schemes over $k$.

Let $\mathcal{M}_m$ be the Lubin-Tate space of level $m$. Letting $m \to \infty$, everything in sight becomes a perfectoid space.

Theorem 8.1.1. The functor $C \to \text{Set}$ that assigns to $R$ the set of isomorphism of classes of pairs $(H, \iota)$, where $H$ is a formal group over $R$, and $\iota : H_0 \to H \otimes_R k$ is an isomorphism, is representable by $A_0 \simeq \text{W}(k)[u_1, \ldots, u_{n-1}]$.

Let $\mathcal{O}_D = \text{End} H_0 \supset \mathbb{Z}_p$. Then $D = \mathcal{O}_D \times \mathbb{Z}_p$ is a division algebra over $\mathbb{Q}_p$, of invariant $1/n$. Since $\mathcal{O}_D$ acts on $H$, it acts on the functor in the above theorem, ence on $\text{W}(k)[u_1, \ldots, u_{n-1}]$. But for $n \geq 2$, this action is very mysterious (e.g. it has not been explicitly been written down).

Drinfeld defined rings $A_0 \to A_1 \to \cdots$, where $A_n$ classifies triples $(H, \iota, \phi)$ over $R$, where $\phi$ is a “level-$n$ structure,” i.e. a certain type of map $(\mathbb{Z}/p^m)^{\oplus n} \to H[p^m]$. Drinfeld level structures are necessary to define the Katz-Mazur models for modular curves over $\mathbb{Z}_p$.

Suppose $n = 1$. Then $H_0 \simeq \hat{G}_m$, and $A_0 = \mathcal{W}(k) = \mathcal{O}_{K_0}$. One has $A_n = \mathcal{W}(k)[\xi_{p^m}] = \mathcal{O}_{K_n}$. The $\{K_m\}$ form the tower of cyclotomic extensions of $K_0$. If $n = 2$, then $H_0 \simeq \hat{E}$, for $E$ a supersingular elliptic curve over $k$. One has $A_m = \mathcal{O}_{X(Np^m), x}$, where $x \in X(Np^m)(k)^{ss}$ is a supersingular point. Drinfeld proved that the $A_m$ are regular local rings admitting an action of both $D^\times$ and $\text{GL}(m)$.

Let’s pass to the generic fiber. Define $\mathcal{M}_{H_0, m}^{(0)} = (\text{Spf} A_m)_{\eta}$. If we allow the $\iota$ to be a quasi-isogeny of height $j$, then we get a different deformation problem, and a space $\mathcal{M}_{H_0, m}^{(j)}$, which is (non-canonically) isomorphic to $\mathcal{M}_{H_0, m}^{(0)}$. (A quasi-isogeny is just an isomorphism in the isogeny category.) For example, $p^{-n}f$ is a quasi-isogeny for all isogenies $f$, but for $n \gg 0$, $p^{-n}f$ is not an honest isogeny. The height of $p^{-n}f$ is $ht(f) - n$.

Let $\mathcal{M}_{H_0, m} = \coprod_{j \in \mathbb{Z}} \mathcal{M}_{H_0, m}^{(j)}$. Then $\mathcal{M}_{H_0, m}$ admits an action of $D^\times$. Put $\mathcal{M}_{H_0, \infty} = \lim_{\leftarrow} \mathcal{M}_{H_0, m}$; for now treat this as a formal projective system. This is called the Lubin-Tate tower. The group $GL_n(\mathbb{Q}_p) \times D^\times$ acts on $\mathcal{M}_{H_0, \infty}$, and this space realizes the local Langlands correspondence.

Jared Weinstein (Feb. 18)
Example 8.1.2 \((n = 1)\). We get \(\mathcal{M}_{H_{0},m}^{(0)} = \text{Spa}(K_{m}, \mathcal{O}_{K_{m}})\), so \(\mathcal{M}_{H_{0},\infty}^{(0)} = \text{Spa}(K_{\infty}, \mathcal{O}_{K_{\infty}})\), at least morally speaking. Here \(K_{\infty} = (\bigcup_{m} K_{m})^{\infty}\); note that this is a perfectoid field. Essentially, this “encompasses” local class field theory.

We would like to give the space \(\mathcal{M}_{H_{0},m}\) a moduli interpretation. Morally, for \((R, R^{+})\) an affinoid \(K_{\infty} = W(k)[\frac{1}{p}]\)-algebra, \(\mathcal{M}_{H_{0},\infty}(R, R^{+})\) is (naively) the set of isomorphism class of triples \((G, \iota, \phi)\), where \(G\) is a formal group over \(R^{+}\), \(\iota : H_{0} \otimes_{k} R^{+}/p \to G \otimes_{R^{+}} R^{+}/p\) is a quasi-isogeny, and \(\phi : \mathbb{Q}_{p}^{\infty} \to VG\) is an isomorphism, where \(V\) is the “\(p\)-adic rational Tate module of \(G\),” i.e. \(VG = (\lim G[p^{n}]) \otimes \mathbb{Q}_{p}\). These triples are only classified up to isogeny. This is not precise because \(R^{+}\) might not be \(p\)-adically complete, and because \(R\) might contain nilpotents. Also, what was written down needs to be sheafified for the topology given by rational subsets (i.e. the topology on \(\text{Spa}(R, R^{+})\)).

Let’s return to the case where the height \(n = 1\). We have \(\mathcal{M}_{H_{0},\infty}(R, R^{+}) = \text{hom}(\mathcal{O}_{K_{\infty}}, R^{+}) = T_{\mathbb{Q}_{p}^{\infty}}(R^{+})^{\text{prim}}\). Here \((-)^{\text{prim}}\) denotes the “set of bases.” We have \(\mathcal{M}_{H_{0},\infty}(R, R^{+}) = V_{\mathbb{Q}_{p}^{\infty}}(R^{+}) \setminus \{0\}\).

8.2 Formal vector spaces

Let \(H\) be a \(p\)-divisible formal group over \(R\), where \(p\) is topologically nilpotent in \(R\). Let \(\tilde{H} = \lim_{\leftarrow p} H\), with the inverse limit being taken in the category of formal schemes. The action of \(p\) on \(\tilde{H}\) is invertible, so \(\tilde{H}\) is a \(\mathbb{Q}_{p}\)-vector space object in the category of formal schemes. We call \(\tilde{H}\) a formal vector space.

Note that \(\mathbb{Q}_{p}/\mathbb{Z}_{p} = \mathbb{Q}_{p}\) (both are abstract groups).

Recall we had a one-dimensional formal group \(H_{0}\) of height 1 over an algebraically closed field \(k\) of characteristic \(p\).

Proposition 8.2.1. We have \(\tilde{H}_{0} \simeq \text{Spf}(k[[T^{1/p^{n}}]])\).

Proposition 8.2.2. Suppose \(R\) is \(p\)-adically complete. Then if \(H\) is a \(p\)-divisible formal group, \(H(R) \to H(R/p)\) is an isomorphism.

Proposition 8.2.3. Let \(H\) be a lift of \(H_{0}\) to \(\mathcal{O}_{K_{0}}\). Then \(\tilde{H} \simeq \text{Spf}(\mathcal{O}_{K_{0}}[[T^{1/p^{n}}]])\).

To see why the proposition is true, compute \(\tilde{H}(R) = \tilde{H}(R/p) \to \tilde{H}_{0}(R/p)\) to see that \(\tilde{H}(R)\) does not actually depend on a choice of lift \(\tilde{H}\). (This is a “crystalline property” of lifts.)

Corollary 8.2.4. Let \(K\) be a perfectoid field containing \(K_{0}\). Let \(\eta = \text{Spa}(K, \mathcal{O}_{K})\). Then \(\tilde{H}_{\eta}^{\text{ad}}\) is a perfectoid space.

In fact, \(\tilde{H}_{\eta}^{\text{ad}}\) is a \(\mathbb{Q}_{p}\)-vector space object in the category of perfectoid spaces over \(K\). Abstractly, \(\tilde{H}_{\eta}^{\text{ad}}\) is the closed unit disk.

Note that \(D \to \text{End} \tilde{H}_{0} \to \text{End} \tilde{H}\). If \((R, R^{+})\) is a perfectoid \(K\)-algebra, \((G, \iota, \phi) \in \mathcal{M}_{H_{0},\infty}(R, R^{+})\), we have \(\phi : \mathbb{Q}_{p}^{\infty} \to VG(R^{+}) = \lim G[p^{n}](R^{+}) \otimes \mathbb{Q}_{p} \to \lim G(R^{+}) \otimes \mathbb{Q}_{p} = G(R^{+}) \simeq \tilde{G}(R^{+}/p) \to \tilde{H}_{0}(R^{+}/p) \simeq \tilde{H}(R^{+})\). So we have a map \(\phi : \mathbb{Q}_{p}^{\infty} \to \tilde{H}(R^{+})\). This gives a morphism \(\mathcal{M}_{H_{0},\infty} \to (\tilde{H}_{\eta}^{\text{ad}})^{\times n}\) which only appears at the infinite level. This morphism is actually an inclusion!
8.3 Connection to p-adic Hodge theory

Start with a $p$-divisible formal group $H_0$ over $k$. Let $M(H_0)$ be its associated Dieudonné module. This is a $W(k)$-module that is free of rank $n = \text{ht}(H_0)$, and has an operators $F,V$ with $FV = p$.

If $R$ is a $k$-algebra, call $R$ $f$-semiperfect if $R = S/I$, where $S$ is a perfect $k$-algebra and $I$ is finitely generated. The main example is $\mathcal{O}_{\mathbb{C}_p}/p \simeq \mathcal{O}_{\mathbb{C}_p}^\flat/p^\flat$.

**Theorem 8.3.1** (Scholze, Weinstein). $\tilde{H}_0(R) \simeq \text{hom}_{F,\phi}(M(H_0), B^+_\text{cris}(R)) \simeq B^+_\text{cris}(R)^{\phi^n = p}$.

Note that $F$ acts on $M(H_0)$ and $\phi$ acts $B^+_\text{cris}(R)$.

As a consequence, $\Lambda^r M(H_0) = M(\Lambda^r H_0)$. Start with $\tilde{H}_0(R)^\times r$. By the theorem, this is isomorphic to

$$\text{hom}(M(H_0)^{\otimes r}, B^+_\text{cris}(R)) \rightarrow \text{hom}(\bigwedge^r M(H_0), B^+_\text{cris}(R)) \rightarrow \Lambda^r H_0(R)$$

(note that not all the maps are isomorphisms).

Recall that we had a morphism $M_{H_0,\infty} \rightarrow (\tilde{H}_\eta)_{ad}^{\times n}$. We also have a morphism $M_{\Lambda^n H_0,\infty} \rightarrow \Lambda^n H_\eta^{ad}$. There is also a morphism $\det : (\tilde{H}_\eta)_{ad}^{\times n} \rightarrow \Lambda^n H_\eta^{ad}$. Finally, there is a morphism $M_{H_0,\infty} \rightarrow M_{\Lambda^n H_0,\infty}$.

**Theorem 8.3.2.** The following diagram is cartesian:

$$
\begin{array}{ccc}
M_{H_0,\infty} & \rightarrow & (\tilde{H}_\eta)_{ad}^{\times n} \\
\downarrow & & \downarrow_{\det} \\
M_{\Lambda^n H_0,\infty} & \rightarrow & \Lambda^n H_\eta^{ad}
\end{array}
$$

All objects carry an action of $GL_n(\mathbb{Q}_p) \times D^\times$. 

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9 The Fargues-Fontaine curve

Let $K$ be a perfectoid extension of $\mathbb{Q}_p$. We can associate to $K$ its tilt $K^p$, and the tilting functor induces an equivalence of categories $K\text{-Perf} \xrightarrow{\sim} K^p\text{-Perf}$. If $F$ is a perfectoid field of characteristic $p$, then there is no canonical choice of some $K$ of characteristic zero with $K^p = F$. Consider the set of isomorphism classes of $(K, \iota)$, where $K$ is perfectoid of characteristic zero, $\iota : F \rightarrow K^p$ is an isomorphism. If we mod out by Frobenius on $F$, then this set is isomorphic to $|X_F|_{\text{deg}=1}$, where $X_F$ is an “algebraic” curve.

9.1 Affine adic curves

Our goal is to construct $X_F^{ad}$. Start with a functor $\text{Perf}_K \rightarrow \text{Adic}_{\mathbb{Q}_p/X_F^{ad}} \rightarrow \text{Perf}_{k(x)}$, the latter functor coming from any $x \in |X_F|_{\text{deg}=1}$. Scholze showed that the composite is an equivalence.

Start with $F$, and $F$ a perfectoid extension of $F_q$, and $E$ some non-archimedean field. Fix $\varpi \in E$ with $|\varpi| < 1$. For $\rho \in (0,1)$, denote by $|\cdot|_\rho$ the unique absolute value on $E$ inducing its topology, such that $|\varpi|_\rho = \rho$.

Let $A$ be a perfectoid $E$-algebra. Let $|\cdot|$ be the spectral norm on $A$. Recall that given a closed interval $I \subset (0,1)$, Fontaine constructed a ring $B_{A,E,I} = B_{E/I}(A)$, which is a “preperfectoid Banach $E$-algebra.” A “preperfectoid $E$-algebra” is just a Banach $E$-algebra that becomes perfectoid after extension to any perfectoid field. Note that if $E'/E$ is an extension, then $B_{A,E,I} \otimes_E E' = B_{A,E',I}$, and if $E$ is perfectoid, then $B_{A,E,I} = B_{A,E',I}$.

Let $(A, A^+)$ be a perfectoid affinoid algebra. There is a natural way to construct a subring $B_{A,E,I}^+ \subset B_{A,E,I}$ coming from $A^+ \subset A^p$.

Definition 9.1.1. $Y_{A,E,I} = \text{Spa}(B_{A,E,I}, B_{A,E,I}^+)$. We will use these spaces to construct the “adic curve.”

If $I \subset I' \subset (0,1)$ and $I = [\rho_1, \rho_2]$, and moreover $\rho_1, \rho_2 \in |F^x|$, say $\rho_1 = |a|$, $\rho_2 = |b|$, then $B_I = B_{I'}(\frac{|a|}{|b|}, \frac{\varpi}{|b|})$. This implies that if $I \subset I'$, then $Y_I \subset Y_{I'}$ is a rational domain.

Definition 9.1.2. $Y_{A,E} = \lim_{\longrightarrow_{I \subset (0,1)}} Y_{A,E,I}$.

This is a preperfectoid space over $E$. (In other words, if $K/E$ is perfectoid, then $Y_{A,E} \otimes K$ is perfectoid.) If $E$ is perfectoid, then $Y_{A,E}^p = Y_{A,E^p}$.

Example 9.1.3. If $E$ has characteristic $p$, then $Y_{A,E}$ is a space over $\text{Spa}(E)$. But $Y_{A,E}$ is also a space over $\text{Spa}(A)$. This is because $B_{A,E,I}$ is an $A$-algebra. Indeed, the Teichmuller representative map $[-] : A \rightarrow B_{A,E,I}$ is additive in this case.

It turns out that in characteristic $p$, we have $Y_{A,E} = \text{Spa}(A) \times_{\text{Spa}F} Y_{F,E}$.

9.2 Gluing

Proposition 9.2.1. Let $f_1, \ldots, f_n, g \in A$, and suppose $(f_1, \ldots, f_n, g) = A$. Then $\sum_i B_I[f_i] + B_I[g] = B_I$.

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Laurent Fargues (Feb. 19)
One also has
\[ B_A\left(\frac{1 - \rho}{\rho}, E, I\right) = B_{A,E,I}\left(\left[\frac{f_1}{g}, \ldots, \frac{f_n}{g}\right]\right) \]
Using this localization property, we can glue affinoid spectra \( Y_{A,E} \) to a functor \( \text{Perf}_F \to \text{PrePerf}_{E/Y_E,F} \), which we will write \( Z \to Y_{Z,E} \).

Let’s consider the action of Frobenius. For \( 0 < \rho < 1 \), write \( \varphi(\rho) = \rho^p \). The Frobenius \( \text{Frob}_q \) acts on \( A \), yielding an isomorphism \( \varphi : B_{A,E,I} \to B_{A,E,I}(\varphi(\rho)) \) of Banach algebras. In terms of Teichmüller expansions, one has
\[
\varphi\left(\sum_{n \geq 0} [x_n] \lambda_n\right) = \sum_{n \geq 0} [x_n^n] \lambda_n
\]

**Theorem 9.2.2.** If the valuation on \( E \) is discrete, and \( \pi_E \) is a uniformizer of \( E \), then
\[
\varphi\left(\sum_{n \geq 0} [x_n] \pi^n_E\right) = \sum_{n \geq 0} [x_n^n] \pi^n_E
\]

Recall that we have a morphism \( Y_{Z,E} \to \text{Spa}(E) \), and there is a Frobenius automorphism \( \varphi \) of \( Y_{Z,E} \). Moreover, \( \varphi(\text{radius } \rho) = \text{radius } \rho^{1/q} \). (Recall that \( 0 < \rho < 1 \).)

**Definition 9.2.3.** \( X_{Z,E}^{ad} = \varphi^Z\backslash Y_{Z,E} \).

This is a preperfectoid space over \( E \). The construction gives us a functor \( \text{Perf}_F \to \text{PrePerf}_{E/X_{Z,E}^{ad}} \). We call \( X_{Z,E}^{ad} \) the “adic curve.”

Suppose \( Z \) is a perfectoid space over \( F \) and \( F \) has characteristic \( p \). The formal construction \( \text{Frob}^Z \backslash Z \) does not actually exist. What we can do is consider \( \varphi^Z\backslash (Z \times_{\text{Spa}_F Y_F} Y_{Z,E}) = \varphi^Z\backslash Y_{Z,F,E} = X_{Z,F} \).

### 9.3 Examples

Suppose \( E \) has characteristic \( p \), e.g., \( E = \mathbb{F}_q((\pi_E)) \). Then we can describe everything in sight. Start by assuming the valuation of \( E \) is discrete, and let \( \pi_E \in E^\circ \) be a uniformizer. Let \( k_E = \mathbb{F}_q \) be the residue field of \( E \). Then \( E = \mathbb{F}_q((\pi_E)) \).

(Throughout, we’ve fixed a finite field \( \mathbb{F}_q \), and an extension \( F \) of \( \mathbb{F}_q \). The residue field \( k_E \) is an extension of \( \mathbb{F}_q \). If we replace \( q \) by \( q^s \), then the new curve is a finite étale cover of the curve defined using \( q \).)

Return to the case \( E = \mathbb{F}_q((\pi_E)) \). Let \( D^*_F \) be the punctured unit disk over \( F \); this is an adic space over \( \text{Spa}(F) \). There is also a morphism \( D^*_F \to D^*_g = \text{Spa}(\mathbb{F}_q((\pi_E))) \). The morphism \( D^*_F \to \text{Spa}(F) \) is locally of finite type, but the morphism \( D^*_F \to D^*_g \) is not locally of finite type.

We can consider \( \varphi^Z\backslash D^*_F \); this does not have a natural structural morphism \( \varphi^Z\backslash \text{Spa}(F) \). On the other hand, it does have a natural morphism \( \varphi^Z\backslash D^*_F \to \text{Spa}(\mathbb{F}_q((\pi_E))) \). The Frobenius \( \varphi \) acts on \( \mathbb{F}_q((\pi_E)) \) by \( \varphi(\sum_{n \in \mathbb{Z}} a_n \pi^n_E) = \sum_{n \in \mathbb{Z}} a_n^{q^n} \pi^n_E \).

If \( E = \mathbb{F}_q((\pi_E^{1/p^\infty})) \), then \( Y_{E,F} = D^*_F \times_{\mathbb{F}_q^{1/p^\infty}} \).

**Example 9.3.1.** Suppose \( E \) is a finite extension of \( \mathbb{Q}_p \). The residue field \( k_E \) of \( E \) is a finite field \( \mathbb{F}_q \). Let \( \mathcal{L}T \) be the Lubin-Tate group law over \( \mathcal{O}_E \). Let \( E_\infty = E(\mathcal{L}T(\pi_E^{q^n})) \). Let \( \pi_E^{q^n} = (\pi_E^{q^n(m)})_{m \geq 0} \); this is a generator of \( T_{\pi_E}(\mathcal{L}T) \). Moreover, \( [\pi_E](\mathcal{L}T(\pi_E^{q^{n+1}})) = \pi_E^{q^n(m)} \). If we
reduce everything modulo $\pi_E$, then $[\pi_E]_{LT}$ reduces to $\text{Frob}_q$, and we have $\text{Frob}_q(\pi_E^{b(m+1)}) = \pi_{E}^{b(m)}$.

In all this, $\pi_{E}^{b} \in E_{\infty} = \mathbb{F}_q((\pi_{E}^{1/p^\infty}))$.

Let $\chi : \text{Gal}(E_{\infty}/E) \to \mathcal{O}_E^\times$ be the Lubin-Tate character. The group $\text{Gal}(E_{\infty}/E)$ acts on $E_{\infty}^\times$, and $(\pi_E^{b})^\sigma = [\chi(\sigma)]_{LT}(\pi_E^{b})$.

Let $G$ be the Lubin-Tate group over $\mathbb{F}_q$, and $\tilde{G} = \lim_{\leftarrow} \times \pi_E^b \times \text{Frob}_q G$. This is a formal $E$-vector space. Finally, put $E = (\tilde{G} \otimes_{\mathbb{F}_q} \mathcal{O}_F)$.

This associates $|\cdot|_{q^{-1/r}}$ with $|\cdot|_{q^{-1/r}}$.

This has implications to $p$-adic Hodge theory. One can consider Huber’s “overconvergent ring” $\mathcal{R}$, which is a union of rings of holomorphic functions on punctured disks.
10 The pro-étale site

For more on the pro-étale site, see Scholze and Bhatt’s paper [BS13], or Scholze’s original paper [Sch13b].

We will not discuss the pro-étale site of a scheme, but the pro-étale site of a locally noetherian adic space \( X \). Let \( X_{\text{ét}} \) be the étale site of \( X \), according to the standard definitions. We will construct a full subcategory \( X_{\text{pro-ét}} \) of the pro-category of \( X_{\text{ét}} \).

10.1 Basic definitions

Definition 10.1.1. The category \( \text{pro}(X_{\text{ét}}) \) has as objects directed inverse systems in \( X_{\text{ét}} \). Morphisms \( U = (U_i) \to V = (V_j) \) are defined by

\[
\text{hom}_X(U,V) = \varprojlim \varinjlim \text{hom}_X(U_i, V_j)
\]

If \( U = (U_i) \in \text{pro}(X_{\text{ét}}) \), then write \( |U| = \varprojlim_i |U_i| \) for the “underlying topological space” of \( U \).

Definition 10.1.2. An object \( U \) of \( \text{pro}(X_{\text{ét}}) \) is in \( X_{\text{pro-ét}} \) if \( U \) can be written as \((U_i)_{i \geq 0}\) such that

1. \( U_0 \to X \) is étale (this is automatic)
2. the transition maps \( U_i \to U_j \) are surjective finite étale

We want to give \( X_{\text{pro-ét}} \) the structure of a site.

Definition 10.1.3. A family of morphisms \( \{f^i : U^i \to U\} \) in \( X_{\text{pro-ét}} \) is a covering if

1. \( f^i \) satisfy the conditions of Definition 10.1.2 translated into “pro-language,” i.e. \( f^i \) is a pro-étale morphism.
2. \( |U| = \bigcup f^i(|U^i|) \).

Example 10.1.4. Start with any diagram \( U = (\cdots \to U_2 \to U_1 \to U_0) \) with the \( U_i \to U_j \) finite étale surjective. Given \( W_{n,n} \to U_n \) étale, we can form \( W_{n,k} = W_{n,n} \times_{U_n} U_k \) for \( k \geq n \). Put \( W^n = (\cdots \to W_{n,n+1} \to W_{n,n}) \); this is an object of \( X_{\text{pro-ét}} \). We have morphisms \( f^n : W^n \to U \) in \( X_{\text{pro-ét}} \), and \( \{f^n\} \) is a covering if and only if \( \{|f^n| : |W^n| \to |U|\} \) is jointly surjective.

Lemma 10.1.5. 1. \( X_{\text{pro-ét}} \) is a site.
2. Pro-étale morphisms are open
3. \( X_{\text{pro-ét}} \) has all finite limits.
4. If \( U \in X_{\text{pro-ét}} \) and \( W \subset |U| \) is a quasi-compact open subset, then \( W = |V| \) for some \( V \to U \) in \( X_{\text{pro-ét}} \).

---

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Proof. We will show that $X_{\text{pro-\acute et}}$ has equalizers (at least, a baby case). Let $a, b : U \to V$ be of the form

$$
\begin{array}{ccc}
& & \\
& & \\
& & \\
& & \\
U_1 & \xrightarrow{=} & V_2 \\
& & \\
& & \\
& & \\
& & \\
U_0 & \xrightarrow{=} & V_0
\end{array}
$$

Complete the diagram as:

$$
\begin{array}{ccc}
& & \\
& & \\
& & \\
& & \\
\vdots & \downarrow & \vdots \\
& & \\
& & \\
& & \\
\vdots & \downarrow & \vdots \\
\text{Eq}(a_1, b_1) & \xrightarrow{=} & U_1 & \xrightarrow{=} & V_2 \\
& & \downarrow & & \downarrow \\
& & \\
\text{Eq}(a_0, b_0) & \xrightarrow{=} & U_0 & \xrightarrow{=} & V_0
\end{array}
$$

What is not clear is that the $E_j \to E_{j-1}$ are surjective. Put $E_{n,k} = \text{im}(E_k \to E_n)$ for $k \geq n$. We can assume everything is affinoid, whence each $E_{n,k}$ is open and closed in $U_n$. Because each $U_n$ has a finite number of connected components, the decreasing chain $E_{n,k} \supset E_{n,k+1} \supset \cdots$ stabilizes. So $E_{n,\infty} = \bigcap_{k \geq n} E_{n,k}$ yields $(E_{n,\infty})_n \in X_{\text{pro-\acute et}}$.

Suppose $f : X \to Y$ is a morphism of locally noetherian adic spaces. Then we get a commutative diagram

$$
\begin{array}{ccc}
X_{\text{pro-\acute et}} & \xleftarrow{f_{\text{pro-\acute et}}} & Y_{\text{pro-\acute et}} \\
\uparrow & & \uparrow \\
X_{\acute et} & \xleftarrow{f_{\acute et}} & Y_{\acute et}
\end{array}
$$

inducing a commutative diagram of associated topoi:

$$
\begin{array}{ccc}
X_{\sim} & f_{\sim} & Y_{\sim} \\
\nu & & \nu \\
X_{\acute et} & f_{\acute et} & Y_{\acute et}
\end{array}
$$

**Lemma 10.1.6.** $H^a(U_{\text{pro-\acute et}}, \nu^* \mathcal{F}) = \lim_{\rightarrow} H^a(U_i, \mathcal{F})$ if $U = (U_i)$ in $X_{\text{pro-\acute et}}$ is quasi-compact and quasi-separated.

**Proof.** Use the Čech-to cohomology spectral sequence. \qed

**Lemma 10.1.7.** $\nu^* Rf_{\acute et, * \mathcal{F}} = Rf_{\text{pro-\acute et}, * \nu^* \mathcal{F}}$.
Example 10.1.8. Let \( X = \text{Spa}(K, K^\circ) \), where \( K \) is a nonarchimedean field. Then \( X_{\text{pro-\acute{e}t}} \) will be the category of profinite sets with continuous \( G_K \)-action. Coverings are families of maps \( \{ f^t : S^t \to S \} \) such that the \( f^t \) are open and jointly surjective. Even when \( K \) is algebraically closed, this is an interesting category.

In this example, \( H^i(X_{\text{pro-\acute{e}t}}, \varprojlim \mathbb{Z}/\ell^n) = H^i_{\text{cont}}(G_K, \mathbb{Z}_\ell) \).

10.2 Perfectoid setting

Let \( K \) be a perfectoid field of characteristic zero. Let \( K^+ \subset K^\circ \) be an open and bounded valuation subring. Let \( X \to \text{Spa}(K, K^+) \) be an adic space.

Definition 10.2.1. \( U \in X_{\text{pro-\acute{e}t}} \) is affinoid perfectoid if \( U \) can be written as \( (U_i) \), where each \( U_i = \text{Spa}(R_i, R_i^+) \), and \( (R, R^+) \) is perfectoid affinoid, where \( R^+ = (\varprojlim R_i^+) \wedge \) and \( R = R^+[\frac{1}{p}] \).

In this setting, \( \text{Spa}(R, R^+) \sim \varprojlim U_i \) in the sense of Scholze.

Example 10.2.2. Let \( T^n \) be the torus \( \text{Spa}(K\langle T_i^{\pm 1}\rangle, K^+(T_i^{\pm 1})) \). Then \( U = \cdots \to T^n \xrightarrow{p} T^n \) is affinoid perfectoid.

Lemma 10.2.3. Let \( U = (U_i) \) be as in Definition 10.2.1. Suppose \( V_{i_0} \to U_{i_0} \) is either finite \( \acute{e}tale \), or a rational subset. Then \( V = (U_i \times_{U_{i_0}} V_{i_0})_{i \geq i_0} \) is also affinoid perfectoid.

Corollary 10.2.4. If \( X \to \text{Spa}(K, K^+) \) is smooth, then every object of \( X_{\text{pro-\acute{e}t}} \) has a covering by affinoid perfectoids.

Proof. We will show that \( X \) has a covering by affinoid perfectoids – this is not a complete proof. The smoothness of \( X \) implies that \( X \) is locally \( \acute{e}tale \) over \( T^n \).

(The argument works in greater generality. An argument of Colmez proves this for general \( X/K \).)

10.3 Contractible objects

Let \( X = \text{Spa}(A, A^+) \) be an affinoid noetherian adic space over \( \text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \). Then there exist “lots” of \( U \in X_{\text{pro-\acute{e}t}} \) such that \( H^i(U, \mathbb{F}_p) = 0 \) for \( i > 0 \).

If \( X \) is connected affinoid noetherian over \( \text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \), then \( H^i_{\text{cont}}(\pi_1(X, \bar{x}), \mathbb{F}_p) \simeq H^i(X_{\acute{e}t}, \mathbb{F}_p) \).

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11 Relative $p$-adic Hodge theory

Everything that follows is joint work with Ruochuan Liu, and is contained in the paper [KL14]. A beamer version of this talk can be found at http://kskedlaya.org/slides/msri2014.pdf.

11.1 Goals of relative $p$-adic Hodge theory

The term “$p$-adic Hodge theory” encompasses two aspects. First, there is external $p$-adic Hodge theory, which compares various cohomology theories (étale, de Rham, crystalline, ...) for varieties over $p$-adic fields. Alternatively, there is a kind of internal $p$-adic Hodge theory, which studies categories of continuous representations of Galois groups of $p$-adic fields (e.g. étale cohomology of varieties).

In this talk, we’ll focus only on the internal aspect of $p$-adic Hodge theory.

Let $K$ be a $p$-adic field (i.e. a field of characteristic zero which is complete for a discrete valuation, with residue field perfect of characteristic $p$). Let $G_K$ be the absolute Galois group of $G_K$. We are interested in the categories $\Rep_{Z_p}(G_K)$ and $\Rep_{Q_p}(G_K)$ of continuous representations of $G_K$ on finitely generated $Z_p$-modules. The latter is the isogeny category of the former.

Let $X$ be an adic space locally of finite type $K$. For $* \in \{Z_p, Q_p\}$, we call an étale $*$-local system a sheaf on $X$ pro-étale locally of the form $Y \mapsto \text{hom}_{\text{cont}}([Y], V)$ for $V$ a finitely generated $*$-module with its standard topology. For example, if $X = \text{Spa}(K, K^\circ)$, then a $*$-étale local system on $X$ is just an object of $\Rep_*(G_K)$, which can be seen by considering the neighborhood $Y = \text{Spa}(C_K, C_K^\circ)$.

In general, étale $Q_p$-local systems are not simply $Z_p$-local systems up to isogeny! There are many natural examples arising from étale covers with noncompact groups of deck transformations, e.g. Tate uniformization of elliptic curves, Drinfeld uniformizations, and Rapoport-Zink period morphisms. Nonetheless, étale $Q_p$-local systems do arise locally from étale $Q_p$-local systems.

11.2 Local systems via perfectoid spaces

Let $K$ be a $p$-adic field. One studies $\Rep_{Q_p}(G_K)$ by passing from $K$ to some sufficiently ramified (strictly arithmetically profinite) algebraic extension $K_\infty$ of $K$. The completion $\widehat{K_\infty}$ is perfectoid, and by tilting + Krasner’s lemma, one has

$$\Rep_{Q_p}(G_{K_\infty}) \simeq \Rep_{Q_p}(G_{\widehat{K_\infty}}) \simeq \Rep_{Q_p}(G_{\widehat{K_\infty}^\flat})$$

To study $\Rep_{Q_p}(G_K)$, you have to add descent data. Generally, we assume $K_\infty/K$ is Galois with $\Gamma = \text{Gal}(\widehat{K_\infty}/K)$ a $p$-adic Lie group, in which case the descent data becomes a $\Gamma$-action.

But descent data can also be viewed as a sheaf condition for the pro-étale topology, in which case we can consider all $K_\infty$ at once! This point of view adapts well to analytic spaces, using perfectoid algebras as the analogue of strictly arithmetically profinite extensions.

For the remainder, let $X$ be an adic space over $Q_p$ which is uniform, in the sense that it is locally Spa($A, A^+$) where $A$ is a $Q_p$-Banach algebra with a submultiplicative norm. In particular, $X$ is reduced. For example, any perfectoid space is uniform. For any adic space $X$ over $Q_p$, there is a unique closed immersed subspace $X^u$ of $X$ which is uniform and has

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the same underlying space. One has $X^u_{\text{ét}} \simeq X_{\text{ét}}$ and $X^u_{\text{pro-ét}} \simeq X_{\text{pro-ét}}$. Adic spaces coming from reduced rigid analytic spaces are uniform. Most of our constructions will not see $A^\times$.

This is related to the fact that $\text{Spa}(A, A^\circ) \to \text{Spa}(A, A^\times)$ is topologically a retraction (when restricted to rank-one points).

Assume $X$ is locally strongly noetherian. As in de Jong’s lecture, we can consider the pro-étale site $X_{\text{pro-ét}}$. This has a structure sheaf

$$\mathcal{O}_X(Y = (Y_i)) = \lim_{\longrightarrow} \mathcal{O}(Y_i).$$

Each $\mathcal{O}_X(Y_i)$ has a power-multiplicative norm (the spectral norm). Recall that $X_{\text{pro-ét}}$ has a basis consisting of affinoid perfectoid opens.

From now on, let $Y$ be an affinoid perfectoid over $X$. From the above, we can define sheaves on $X_{\text{pro-ét}}$ by defining them on such $Y$.

**Definition 11.2.1.** The sheaf $\hat{\mathcal{O}}_X$ on $X_{\text{pro-ét}}$ is defined by $\hat{\mathcal{O}}_X(Y) = \text{completion of } \mathcal{O}(Y)$ for the spectral norm, when $Y$ is as above.

**Definition 11.2.2.** Let $\overline{\mathcal{O}}_X(Y) = \hat{\mathcal{O}}_X(Y)^\flat$, for $Y$ as above.

Recall that for $R$ a perfect ring of characteristic $p$, there is a ring $W(R)$ of $p$-typical Witt vectors. Let $[-] : R \to W(R)$ be the Teichmuller character.

**Definition 11.2.3.** Define $\overline{A}_X$ on $X_{\text{pro-ét}}$ by $\overline{A}_X(Y) = W(\overline{\mathcal{O}}_X(Y))$ for $Y$ as above.

It turns out that if $R$ has a power-multiplicative norm, then for any $r > 0$, the set $W^r(R) \subset W(R)$ consisting of $\sum p^n [x_n]$ with $\lim_{n \to \infty} p^n n|x_n|^r = 0$ is a subring.

**Definition 11.2.4.** Define $\overline{A}_X^{1/r}$ on $X_{\text{pro-ét}}$ for $\overline{A}_X^{1/r}(Y) = W^r(\overline{\mathcal{O}}_X(Y))$ for $Y$ as above. Put $\overline{A}_X = \lim_{\longrightarrow_{r \to 0^+}} \overline{A}_X^{1/r}$.

Let $S$ be a ring and $\varphi$ an automorphism of $S$. A $\varphi$-module on $S$ is a finite projective $S$-module $M$ together with an isomorphism $\varphi^* M \simeq M$.

**Theorem 11.2.5** (Katz, SGA 7). Let $R$ be a perfect $\mathbb{F}_p$-algebra. Then the following categories are equivalent:

1. étale $\mathbb{Z}_p$-local systems on $\text{Spec}(R)$
2. $\varphi$-modules over $W(R)$
3. $\varphi$-modules over $W^1(R) = \bigcup_{r > 0} W^r(R)$

(The last category does not appear in SGA 7). For $R = F$ a field, the functors between $\mathbb{Z}_p$-representations of $G_F$ and $\varphi$-modules over $W(F)$ are

$$V \mapsto (V \otimes W(F))^G_F$$

$$M \mapsto (M \otimes_{W(F)} W(F))^{\varphi = 1}$$

A $\varphi$-module over a ring sheaf $\ast_X$ on $X_{\text{pro-ét}}$ is a “quasicoherent finite projective” sheaf $\mathcal{F}$ of $\ast_X$-modules together with an isomorphism $\varphi^* \mathcal{F} \simeq \mathcal{F}$. 

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Proposition 11.2.6. Quasicoherent finite projective modules over $\tilde{A}_X|_Y$ or $\tilde{A}^\dagger_X|_Y$ correspond to finite projective modules over $\tilde{A}_X(Y)$ or $\tilde{A}^\dagger_X(Y)$, respectively. Also, these sheaves are acyclic.

The following is a “sheafified Artin-Shreier theorem.”

Theorem 11.2.7. The following categories are equivalent:

1. étale $\mathbb{Z}_p$-local systems on $X$
2. $\varphi$-modules over $\tilde{A}_X$
3. $\varphi$-modules over $\tilde{A}^\dagger_X$

The functors are

$$T \mapsto T \otimes_{\mathbb{Z}_p} \tilde{A}_X$$
$$M \mapsto M^{\varphi=1}$$

(these are between étale $\mathbb{Z}_p$-local systems and $\varphi$-modules over $\tilde{A}_X$).

Theorem 11.2.8. For $T$ an étale $\mathbb{Z}_p$-local system on $X$ corresponding to a $\varphi$-m module $\mathcal{F}$ over $\tilde{A}_X$ and a $\varphi$-m module $\mathcal{F}^\dagger$ over $\tilde{A}^\dagger_X$, the sequences

$$0 \to T \to \mathcal{F}^* \to \mathcal{F}^* \to 0$$

are exact for $* \in \{\emptyset, \dagger\}$.

The point is that $\mathcal{F}$ is acyclic on every perfectoid subdomain, not just sufficiently small ones. This recovers Herr’s formula for Galois cohomology of $\mathbb{Z}_p$-local systems over a $p$-adic field.

Morally, this comes down to $W^\dagger(R)^{\varphi=1} = W^\dagger(R^{\varphi=1})$, where $R^{\varphi=1} = \text{hom}_{\text{cont}}(\text{Spec } R, \mathbb{F}_p)$.

11.3 Robba rings and $\mathbb{Q}_p$-local systems

A $= \lim_{\leftarrow} \mathbb{Z}/p^n((\pi))$; this is a Cohen ring with residue field $\mathbb{F}_p((\pi))$.

$A^{1,r}$ are elements of the previous guy that converge for $p^{-r} \leq |\pi| < 1$.

$A^\dagger = \bigcup_{r \geq 0} A^{1,r}$.

$B^\ast = A^\ast[p^{-1}]$ for $* \in \{\emptyset, \dagger, r\}$

$C^{[s,r]}$ is analytic functions on $p^{-r} \leq |\pi| \leq p^{-s}$

$C^r$ is analytic functions on $p^{-r} \leq |\pi| < 1$.

...
12 \( p \)-adic Hodge theory for rigid spaces

We will be discussing the finiteness of étale cohomology modulo \( p \), and some basic comparison theorems.

12.1 Comparison theorem

The main theorem is the following:

Theorem 12.1.1. Let \( C \) be an algebraically closed complete extension of \( \mathbb{Q}_p \). Let \( X \) be a proper smooth adic space over \( \text{Spa}(C, \mathcal{O}_C) \). Let \( \mathcal{L} \) be an \( \mathbb{F}_p \)-local system on \( X \). Then

1. \( H^i(X_{\text{ét}}, \mathcal{L}) \) is a finite-dimensional \( \mathbb{F}_p \)-vector space for all \( i \geq 0 \)
2. there is an almost isomorphism (of \( \mathcal{O}_C \)-modules)

\[
H^*(X_{\text{ét}}, \mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \simeq H^*(X_{\text{ét}}, \mathcal{L} \otimes \mathcal{O}_X^+/p).
\]

There is a relative version of this theorem, which follows from the base change theorem and a slightly more general version of the main theorem. Moreover, this theorem implies the \( \mathcal{B} \)\text{-}\text{dR}-comparison theorem.

The proof works by reduction to coherent cohomology via Artin-Schreier sequences. This involves tilting and the pro-étale site. Recall that the pro-étale site \( X_{\text{pro-ét}} \) computes étale cohomology via \( \nu : X_{\text{pro-ét}} \rightarrow X_{\text{ét}} \). Here we are mapping \( \mathcal{F} \) on \( X_{\text{ét}} \) to \( \mathcal{R}\nu_* \mathcal{F} \). Over \((K, K^+)\), affinoid perfectoids form a basis for the pro-étale topology. Finally, if \( U \) is an affinoid perfectoid, then \( H^i(U, \mathcal{O}_X^+/p) \) is almost zero for \( i > 0 \). One does this by relating this cohomology group to \( H^i(W_{\text{ét}}, \mathcal{O}_X^+/p) \) for \( W \) an affinoid perfectoid space, and applying the almost version of Tate’s acyclicity theorem.

We will use a twisted version of the above theorem.

Theorem 12.1.2. Let \( \mathcal{L} \) be an étale \( \mathbb{F}_p \)-local system. Then \( H^i(U, \mathcal{L} \otimes \mathcal{O}_X^+/p) \) is almost zero for \( i > 0 \). Moreover, \( H^i(U, \mathcal{L} \otimes \mathcal{O}_X^+/p) = M(U) \), for \( \tilde{U} = \text{Spa}(S, S^+) \), is a finitely generated projective \( S^+ \)-module compatible with base change.

We want to use the classical Artin-Schreier sequence on \( X_{\text{pro-ét}} \). We have an exact sequence:

\[
0 \rightarrow \mathcal{L} \rightarrow \mathcal{L} \otimes \mathcal{O}_X^+/p \xrightarrow{\varphi^{-1}} \mathcal{L} \otimes \mathcal{O}_X^+/p \rightarrow 0
\]

To see exactness, look locally at some affinoid perfectoid \( U \in X_{\text{pro-ét}} \). We can assume \( \mathcal{L}_U \) is trivialized, so it is enough to show surjectivity. To see that, we use tilting. Let \( \pi \in \mathcal{O}_{\tilde{U}}^+ \) be such that \( \pi^2 = p \).

(If \( U = (U_i = \text{Spa}(A_i, A_i^+))) \in X_{\text{pro-ét}}, \) then \( \tilde{U} \) comes from the completion of \( \text{lim} \ A_i \).)

The map \( \varphi^{-1} : \mathcal{O}_{\tilde{U}}^+/p \rightarrow \mathcal{O}_{\tilde{U}}^+/p \) is étale, and is the same as \( \mathcal{O}_{\tilde{U}}^+/p \rightarrow \mathcal{O}_{\tilde{U}}^+/\pi \). Pass back from the tilted site using the equivalence \( \tilde{U}_{\text{ét}} \simeq \tilde{U}_{\text{ét}} \). We get

\[
\cdots \rightarrow H^i(X_{\text{ét}}, \mathcal{L}) \rightarrow H^i(X_{\text{ét}}, \mathcal{L} \otimes \mathcal{O}_X^+/p) \rightarrow H^i(X_{\text{ét}}, \mathcal{L} \otimes \mathcal{O}_X^+/p) \rightarrow \cdots
\]

\[
\mathbb{F}_p \rightarrow (\mathcal{O}_C/p)^r \rightarrow (\mathcal{O}_C/p)^r \rightarrow 0
\]

---

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The isomorphism uses the fact (which we will prove later) that $H^j(X_{ét}, \mathcal{L} \otimes \mathcal{O}_X^+/p)$ is an almost finitely generated $\mathcal{O}_C/p$-module, as well as some facts about Frobenius. The vertical isomorphisms are only almost isomorphisms, and if we knew the appropriate finiteness conditions, we could pass from almost mathematics to “real” mathematics.

Since we are trying to prove a finiteness theorem, we instead use a modified form of Artin-Schreier theory to get rid of the “almost” part. We have an exact sequence

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{L} \otimes \hat{O}_X, \quad \mathcal{L} \otimes \hat{O}_X, \rightarrow 0$$

where $\hat{O}_X = \lim_{\leftarrow p} \mathcal{O}_X^+/p$ and $\hat{O}_X = \hat{O}_X \otimes \mathcal{O}_C^+ C^\flat$. Passing to cohomology, we get

$$\cdots \rightarrow H^j(X_{pro-ét}, \mathcal{L}) \rightarrow H^j(X_{pro-ét}, \mathcal{L} \otimes \hat{O}_X^+) \rightarrow H^j(X_{pro-ét}, \mathcal{L} \otimes \hat{O}_X^+) \rightarrow \cdots$$

$$0 \rightarrow \mathcal{F}_p \rightarrow (C^\flat)^r \rightarrow (C^\flat)^r \rightarrow 0$$

The proof of Theorem 12.1.1 is nearly complete. It remains to show that $H^j(X_{ét}, \mathcal{L} \otimes \mathcal{O}_X^+/p) = 0$ for $j > 2 \dim X$. Put $d = 2 \dim X$. Look at the projection $\varepsilon : X_{ét} \rightarrow X_{an}$. Since the dimension of $X_{an}$ is also $d$, it suffices to show that $R^j \varepsilon_* (\mathcal{L} \otimes \mathcal{O}_X^+/p)$ are almost zero for $j > d$. We will assume $X$ is affinoid, smooth, and has a “good” coordinate system $X \rightarrow T^d$, where $T^d$ is a $d$-dimensional torus, and that this map is the composition of a rational endomorphism with a finite étale map.

Write $T^d = \text{Spa}(C(T_i^{+1}), \mathcal{O}_C(T_i^{+1}))$. Let $T/T$ be the affine perfectoid, $\bar{T} = \lim_{\leftarrow m} T_m$, where $T_m = \text{Spa}(C(T_1^{+1/p^m}), \mathcal{O}_C(T_i^{+1/p^m}))$. Let $\bar{X} = X \times_T \bar{T}$. This is a pro-covering of $X$.

(Somewhere in here, we put $T^d = T$.) Then

$$R\Gamma(X_{pro-ét}, \mathcal{L} \otimes \mathcal{O}_X^+/p) = \mathcal{E}(\bar{X}/X, R\Gamma(\bar{X}_{pro-ét}^\bullet, \mathcal{L} \otimes \mathcal{O}_X^+/p))$$

But

$$H^j(\bar{X}_{pro-ét}^\bullet, \mathcal{L} \otimes \mathcal{O}_X^+/p) = H^j(\bar{X} \times Z_p^{d(i-1)}, \mathcal{L} \otimes \mathcal{O}_X^+/p)$$

$$= \text{hom}_{cont}(\mathbb{Z}_p^{d(i-1)}, H^j(\bar{X}, \mathcal{L} \otimes \mathcal{O}_X^+/p))$$

The $H^j(\bar{X}, \mathcal{L} \otimes \mathcal{O}_X^+/p)$ are almost zero for $j > 0$. It follows that $H^j(X_{pro-ét}, \mathcal{L} \otimes \mathcal{O}_X^+/p)$ is almost isomorphic to $H^j_{cont}(\mathbb{Z}_p^d, M)$, where $M$ is some finitely generated projective $S^\flat$-module.

The group $\mathbb{Z}_p^d$ has cohomological dimension $d$, so everything vanishes (in the almost sense) if $j > d$.

### 12.2 Proof of auxiliary result

Let’s prove the following claim. Let $X$ be a proper smooth adic space over $\text{Spa}(K, \mathcal{O}_K)$, where $K$ contains all $p$-power roots of unity and has characteristic zero. Then $H^j(X_{ét}, \mathcal{L} \otimes \mathcal{O}_X^+/p)$ is a finitely generated $\mathcal{O}_K$-module. The idea is to mimic the proof of finite-generation of coherent cohomology for rigid analytic spaces. Write $X = \bigcup V_i$, where the $V_i$ are affinoids.
with good coordinates. We also want a covering $X = \bigcup V_i',\text{ where each } V_i' \subseteq V_i$. This gives us two spectral sequences

\[ E_1^{m_1 m_2} = \bigoplus_{|j|=m_1+1} H^m(V_{i,\text{ét}}, L \otimes O_X^+/p) \Rightarrow H^{m_1+m_2}(X, L \otimes O_X^+/p) \]

\[ E_2^{m_1 m_2}(V) = \bigoplus H^i(X, V_j', L \otimes O_X^+/p) \Rightarrow H^{m_1+m_2}(X,-) \]

It suffices to show that the image of the restriction map is a finitely generated $O_X$-module.

**Lemma 12.2.1.** Let $V$ be a “good” affinoid, $V' \subseteq V$ a rational subset. Then the image of $\alpha = \text{res} : H^i(V_{\text{ét}}, L \otimes O_X^+/p) \rightarrow H^i(V'_{\text{ét}}, L \otimes O_X^+/p)$ is an almost finitely generated $O_K$-module.

**Proof.** Write $V' \hookrightarrow V \rightarrow T$, where $T$ is a torus. Let $\widehat{T}$ be the “perfectoidification of $T$.” This induces covers $\widetilde{V} \rightarrow V$ and $\widetilde{V}' \rightarrow V'$. Let $\widetilde{V} = \text{Spa}(S', S'^+)$ and $\widetilde{V} = \text{Spa}(S, S^+)$ be the associated adic spaces coming from completion. (Formally, $\widetilde{T} = \varprojlim \text{spa}(T^1/p^m)$ and $\widehat{T}$ is the completion.

First we pass to the group coordinates, giving

\[ \alpha : H^i_{\text{cont}}(\mathbb{Z}_p^d, M) \rightarrow H^i_{\text{cont}}(\mathbb{Z}_p^d, M \otimes S^+/p S'^+ / p). \]

We can assume $M = S^+/p$. Next, we consider $(S_m, S_m^+), (S'_m, S'_m^+)$, which are the “levels” in the pro-étale cover $\widetilde{T} \rightarrow T$. The image of

\[ H^i_{\text{cont}}(\mathbb{Z}_p^d, (S_m^+ \otimes R_m^+R^+)/p) \rightarrow H^i_{\text{cont}}(\mathbb{Z}_p^d, S'_m \otimes \cdots) \]

is finitely generated over $O_K$.

The rest of the proof uses Hochschild-Serre to reduce to a map $S_m^+ \otimes H^i \rightarrow S_m' \otimes H^i$. \qed
13 Lubin-Tate spaces 2

Recall from last time, we started with a $p$-divisible group $H_0$ over an algebraically closed field $k$ of characteristic $p$. There is an associated Dieudonné module $M(H_0)$, a free module over $W(k)$ with rank the height of $H_0$. There are also endomorphisms $F, V$ of $M$ with $FV = p$. The dimension of $H_0$ is $\dim_k(M/FM)$.

We tried to make $\bigwedge^n M(H_0)$ into a Dieudonné module. We give it Frobenius $\bigwedge^n F$, but for $\bigwedge^n H_0$ to actually be a Dieudonné module, we need $\dim H_0 \leq 1$. If $\dim H_0 = 1$, we can form $\bigwedge^r H_0$, which has $h^t H_0 = \binom{n}{r}$ and $\dim H_0 = \binom{n-1}{r-1}$. (Here $n = \dim H_0$.) If $H_0$ has dimension one and height $n$, then $F$ has determinant $p$, so $\bigwedge^n H_0 \simeq \mu_{p^n}$.

From last time, there is a Cartesian diagram

$$
\begin{array}{ccc}
M_{H_0, \infty} & \longrightarrow & (\tilde{H}_\eta)^n \\
\downarrow & & \downarrow \det \\
M_{\bigwedge^n H_0, \infty} & \longrightarrow & \bigwedge^n H^\text{ad}_\eta
\end{array}
$$

where $M_{\bigwedge^n H_0, \infty} \simeq \nu_{p^n} \setminus \{0\}$, and everything involved is a perfectoid space. The point $\eta$ is $\text{Spa}(K)$, for $K$ some perfectoid field containing $W(k)$.

If we had used $\eta = \text{Spa}(K_0, O_{K_0})$, everything would be a preperfectoid space, and $M_{\bigwedge^n H_0, \infty} = \prod_\mathbb{Z} \text{Spa}(K_\infty, O_{K_\infty})$, where $K_\infty = \hat{K}_0(\mathbb{G}_m)$. We have

$$(\tilde{H}_\eta)^n \simeq \text{Spf} \left( O_{K_0} \left[ X_1^{1/p^n}, \ldots, X_n^{1/p^n} \right] \right)^{\text{ad}}$$

$$(\bigwedge^n H_\eta) \simeq \text{Spf} \left( O_{K_0} \left[ T^{1/p^n} \right] \right)^{\text{ad}}$$

so the “determinant map” comes from a “generalized power series” $\delta(X_1, \ldots, X_n)$ in $X_1^{1/p^n}, \ldots, X_n^{1/p^n}$.

On the level of $\mathbb{C}_p$-points, this diagram is due to Fargue’s book on the Lubin-Tate tower.

13.1 Some explicit formulas

Let $R$ be an $f$-semiperfect $k$-algebra. Then

$$\tilde{H}_0(R) \simeq \text{hom}_{F, \phi}(M(H_0), B^{+}_{\text{cris}}(R))$$

If $n = 1$, then the map is

$$R^\phi \supset \lim_{x \to x^p} (1 + \text{Nil}(R)) = \tilde{\mu}_{p^n}(R) \to B^{+}_{\text{cris}}(R)^{\phi=p}$$

$$x \mapsto \log[x]$$

Let’s return to the general case. Let $H_0$ be a one-dimensional formal group of height $n$, and let $H$ be a lift of $H_0$ to $O_{K_0} = W(k)$. Let $\log_H : H \otimes K_0 \to \hat{\mathbb{G}}_a$ be an isomorphism. Given a power series $g(T) \in K_0[[T]]$, define $\delta g(X, Y) = g(X + H Y) - G(X) - G(Y)$. Note that $\delta \log_H = 0$.

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Define
\[ \text{Qlog}(H) = \{ g \in TK_0[[T]] : dg, \delta g \text{ are integral} \}/T\mathcal{O}_{K_0}[[T]] . \]

It turns out that \( M(H_0) \simeq \text{Qlog}(H) \), and is spanned by \( \log_H(T), \ldots, \log_H(T^{p^{n-1}}) \) (at least up to \( 1/p \)). Write \( e = \log_H(T), \ldots, F^{n-1}e = \log_H(T^{p^{n-1}}) \).

(A good place to learn about this is the Gross-Hopkins’s paper, or Katz’s paper on Crystalline cohomology, Jacobi sums, . . .).

Given \( g \in \text{Qlog}(H_0) \), we can evaluate \( g \) on \( \tilde{H} \). If \( (R, R^+) \) is a \( K_0 \)-algebra, then we get \( g : \tilde{H}(R^+) \to R \), given by \( (x_0, x_1, \ldots) \mapsto \lim_{m \to \infty} p^m g(x_m) \).

The map \( g \) is actually a homomorphism \( \tilde{H} \to \mathbb{G}_a \). This gives
\[
\begin{array}{ccc}
\tilde{H}_0(R^+) & \cong & [M(H_0)^* \otimes B_{\text{cris}}^+(R^+)]^{F \otimes \phi} \\
\text{qlog}_{H_0} & \downarrow & \downarrow \otimes \theta \\
M(H_0)^* \otimes R & & \\
\end{array}
\]

the “quasi-logarithm map” is \( X \mapsto (g \mapsto g(X)) \), under the isomorphism \( M(H_0)^* \otimes R = \text{hom}(M(H_0), R) \). So we get
\[
\begin{array}{ccc}
(H_0^{ad})^n & \xrightarrow{\text{qlog}_{H_0}} & (M(H_0)^*)^n \otimes \mathbb{G}_a \simeq \mathbb{G}_a^2 \\
\det & \downarrow & \downarrow \det \\
\bigwedge^n H_\eta & \xrightarrow{\text{qlog}_{H_0}^\text{ad}} & \bigwedge^n M(H_0)^* \otimes \mathbb{G}_a \simeq \mathbb{G}_a \\
\end{array}
\]

Choose coordinates on \( H \) and \( \bigwedge^n H \) so that
\[
\log_H(T) = T + \frac{T^p}{p} + \frac{T^{p^2}}{p^2} + \cdots
\]
\[
\log_H^n(T) = T + (-1)^{n-1} \frac{T^p}{p} + \frac{T^{p^2}}{p^2} + \cdots
\]

**Proposition 13.1.1.** In
\[
\begin{array}{cc}
\text{Spf}(\mathcal{O}_{K_0}[[T^{1/p^\infty}]])^\text{ad} & \xrightarrow{\log_{H_\eta}^\text{ad}} & H_\eta^\text{ad} \\
\log_{H_\eta} & \downarrow & \downarrow \log_{H_\eta} \\
& & \mathbb{G}_a \\
\end{array}
\]

the “downward-right map” is
\[
x \mapsto \cdots + p^2 x^{1/p^{2n}} + px^{1/p} + x + \frac{x^{p^n}}{p} + \frac{x^{p^{2n}}}{p^2} + \cdots = \sum_{i \in \mathbb{Z}} \frac{x^{p^i}}{p^i}
\]

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Proposition 13.1.2. We have
\[ d(X_1, \ldots, X_n) = (\bigwedge H) \sum_{(a_1, \ldots, a_n) \in \mathbb{Z}^n} \varepsilon(q) X_1^{a_1} \cdots X_n^{a_n} \]
where \( \varepsilon \) is the sign of \( i \mapsto a_{i+1} \mod n \).

Proof. Plug into the previous diagram and see that it commutes. \( \square \)

There is a generalization for “formal \( F \)-vector spaces,” where \( F \) is a finite extension of \( \mathbb{Q}_p \).

13.2 Period maps
Recall the quasi-logarithm \( \text{qlog}_H : \tilde{H}^\text{red} \to M(H_0)^\star \otimes \mathbb{G}_a \). Recall that \( M_{H_0, \infty}(R, R^+) \) is the set of tuples \( (x_1, \ldots, x_n) \in \tilde{H}(R^+)^n \) such that \( d(x_1, \ldots, x_n) \in V_{\mu_p} \setminus \{0\} \). Alternatively, this is tuples \( (x_1, \ldots, x_n) \) linearly independent over \( \mathbb{Q}_p \), such that \( 0 = \log \delta(x_1, \ldots, x_n) = \det(\text{qlog}(x_1)) \).

We have elements \( \text{qlog}(x_1), \ldots, \text{qlog}(x_n) \) in \( M(H_0)^\star \otimes \mathbb{G}_a \). The span of these elements gives an element of \( \mathbb{P}^{n-1} \). We could also consider the vector of linear relations; this lives in \( \mathbb{P}^{n-1} \setminus \) all \( \mathbb{Q}_p \)-rational hyperplanes. This is the Drinfeld upper half-plane \( \Omega \).

This is the “Gross-Hopkins period mapp” \( \pi_{\text{GH}} : M_{H_0, \infty} \to \mathbb{P}^{n-1} = \mathbb{P}(M(H_0)) \), and it corresponds to quotiening out by the action of \( GL_n(\mathbb{Q}_p) \). The “Hodge-Tate period map” \( \pi_{\text{HT}} : M_{H_0, \infty} \to \Omega \) corresponds to quotienting out by \( D^\times \). Things live inside a Shimura variety \( Sh_{\infty} \), and we have a commutative diagram

\[ \begin{array}{ccc}
Sh_{\infty} & \xrightarrow{\pi_{\text{HT}}} & \mathbb{P}^{n-1} \\
\downarrow & & \downarrow \\
\Pi M_{H_0, \infty} & \rightarrow & \Omega
\end{array} \]

13.3 Some conjectures
We are interested in the geometry of \( M_{H_0, \infty} \) over \( \eta = \text{Spa}(C, \mathcal{O}_C) \) for some complete algebraically closed field \( C/\mathbb{Q}_p \). We have a map \( M_{H_0} \to V_{\mu_p} \setminus \{0\} \). Let \( M_{H_0, \infty} \) be the fiber over some geometric point. The idea is that \( H^\star(M_{H_0, \infty}, \mathbb{Q}_\ell) \) should realize the local Langlands correspondance. For some \( \pi \subset H^\star \) that is supercuspidal, we know that it is of the form \( \text{ind}_{\mathcal{X}}^{\text{GL}_n(\mathbb{Q}_p)} \tau \), where \( \mathcal{X} \) is some open compact=mod-center, and \( \tau \) is finite-dimensional.

Let \( (\mathbb{P}^{n-1})_{\text{special}} \subset \mathbb{P}^{n-1} \) be the locus where the stabilizer inside \( P\text{GL}_n(\mathbb{Q}_p) \) is nontrivial. Let \( (\mathbb{P}^{n-1})_{\text{nonsp}} \subset \Omega \) be its complement. Let \( M_{H_0, \infty}^{\text{nonsp}} = \pi_{\text{HT}}^{-1}(\mathbb{P}^{n-1})_{\text{nonsp}} \).

Conjecture 13.3.1. The space \( M_{H_0, \infty}^{\text{nonsp}, \star} \) can be covered by affinoids \( U \) such that \( \dim H^\star(U, \mathbb{Q}_\ell) < \infty \).

This is known for \( n = 2 \).

The following is more of a fantasy than a conjecture: \( M_{H_0, \infty}^{\text{nonsp}, \star} \) is locally \( p \)-finite.
14 Shimura varieties and perfectoid spaces 1: completed cohomology

This is a report on Scholze’s preprint [Sch13a].

14.1 Some definitions

Let’s start with $G$, the $\mathbb{Z}_p$-points of a some algebraic group. The group $G$ has a natural filtration $G_r$, where $G_r$ is the level $p^r$ congruence subgroup of $G$.

Let $X$ be a manifold with a tower $X_r$ over $X$. We require that $G/G_r$ acts on each $X_r$, making it a principal homogeneous space over $X_r$.

**Example 14.1.1.** We could have $G = \mathbb{Z}_p$, and have $G$ act on the tower $\cdots \to S^1 \to S^1 \to \cdots \to S^1$, with each map being “raise to $p$-th power.”

**Example 14.1.2.** Let $X = Y(N)$ be a modular curve, $X_r = Y(Np^r)$, and $G_r = \text{GL}_2(\mathbb{Z}/p^r)$. The group $G$ is just $\text{GL}_2(\mathbb{Z}_p)$.

In this context, one can define the completed cohomology of the tower by

$$\tilde{H}^i(X_\bullet, \mathbb{Z}_p) = \lim_{\leftarrow s} \lim_{\rightarrow r} H^i(X_r, \mathbb{Z}/p^s)$$

An important thing to notice is that these completed cohomology groups are $p$-adically complete, and “see $p$-power torsion.” Also, $p$-power torsion at the finite level can “patch together” to yield torsion-free cohomology classes at the infinite level.

**Example 14.1.3.** Go back to $G = \mathbb{Z}_p$ and $X_1 = S^1$. Then $\tilde{h}^0 = \mathbb{Z}_p$ and $\tilde{h}^1 = 0$.

**Example 14.1.4.** Here let $X_r = Y(Np^r)$ and $G = \text{GL}_2(\mathbb{Z}_p)$. One gets $\tilde{h}^0 = \mathbb{Z}_p \left[ (\mathbb{Z}/N)^\times \times \mathbb{Z}_p^\times \right]$ and $\tilde{h}^1$ is interesting in the context of $p$-adic Langlands.

It is natural to ask about how the relation between this completed cohomology and cohomology at finite level. This is given by a Hochschild-Serre spectral sequence:

$$E_2^{i,j} = H^i(G_r, \tilde{H}^j) \Rightarrow H^{i+j}(X_r, \mathbb{Z}_p).$$

For the tower of circles, $H^0(\mathbb{Z}_p, \mathbb{Z}_p) = H^1(\mathbb{Z}_p, \mathbb{Z}_p) = \mathbb{Z}_p$. So some part of the cohomology comes from the cohomology of a $p$-adic Lie group, but that part “dies” when taking completed cohomology. This often lets one restrict to looking at “Hecke eigenvalues on completed cohomology.”

If $W$ is a free $\mathbb{Z}_p$-module of finite rank, with a continuous $G$-action, then we get compatible local systems $\mathcal{W}_r$ over each $X_r$. We have a spectral sequence

$$E_2^{ij} = \text{Ext}_{\mathbb{Z}_p[G_r]}(W^\vee, \tilde{H}^j) \Rightarrow H^{i+j}(X_r, \mathcal{W}_r).$$

So somehow completed cohomology packages all the various choices of weight and level into a single object.
14.2 The setting

Let $G$ be a reductive group over $\mathbb{Q}$. Let $K_f$ be an open compact subgroup of $G(\mathbb{A}^f)$. Write $Y(K_f)$ for the quotient

$$Y(K_f) = G(\mathbb{Q}) \backslash G(\mathbb{A}) / A_\infty^0 K_\infty^0 K_f,$$

where $A_\infty^0$ is the connected component of $\mathbb{R}$-points of the maximal $\mathbb{Q}$-split torus in the center of $G$, and $K_\infty^0$ is the connected component of the maximal compact subgroup of $G(\mathbb{R})$.

**Example 14.2.1.** Suppose $G = GL(2)$. Then $A_\infty^0 = \mathbb{R}_>^\times$ and $K_\infty^0 = SO(2)$. Thus the quotient $GL_2(\mathbb{R})/A_\infty^0 K_\infty^0 = \mathbb{C} \smallsetminus \mathbb{R}$. It turns out that $Y(K_f)$ is a modular curve of some type.

Suppose $G$ is $GL(2)$ over some imaginary quadratic field. Then $G(\mathbb{R}) = GL_2(\mathbb{C})$, $A_\infty^0 = \mathbb{R}_>^\times$, $K_\infty^0 = U(2)$, and

$$GL_2(\mathbb{C})/A_\infty^0 K_\infty^0 = PSL_2(\mathbb{C})/SO(3) = \mathbb{H}^3,$$

which is hyperbolic 3-space.

The manifolds $Y(K_f)$ are not generally algebraic varieties – they are only manifolds. This makes it especially surprising that one can attach Galois representations to torsion classes in their cohomology.

**Theorem 14.2.2** (Franke). $H^i(Y(K_f), \mathbb{C})$ is “computed by automorphic forms.”

This is a generalized Eichler-Shimura theory.

Write $\dim Y(K_f) = 2q_0 + \ell_0$, where

$$\ell_0 = \text{rk}(G) - \text{rk}(A_\infty^0) - \text{rk}(K_\infty^0)$$

$$q_0 = ?$$

We know that for algebraic varieties, the “most interesting cohomology” occurs in “middle degree.” In some sense, $H^{q_0}$ is the “first interesting degree” in the cohomology (with $\mathbb{C}$-coefficients).

Fix a ground level $K_f = K_p^p K_p$. (Here $K_p^p$ is a “prime-to-$p$-part” and $K_p$ is a “$p$-part.”) We’ll fix $K^p$ and vary $K_p$, to get a tower $Y(K^p K_p)$. The group $G(\mathbb{Q}_p)$ acts on this tower.

Form $\tilde{H}^i$ and $\tilde{H}^i_\ell$; both of these admit an action of $G(\mathbb{Q}_p)$. One would hope that these are described by $p$-adic local Langlands. Let $\mathcal{T}$ be the algebra generated by “Hecke operators of level $\ell \nmid pN$,” where $N$ is the level of $K^p$.

**Conjecture 14.2.3** (Calegari, Emerton). $\tilde{H}^i = 0$ if $i > q_0$.

**Theorem 14.2.4** (Scholze). The conjecture is true for many Shimura varieties.

[...stopped writing on board, drew picture of $Y_0(11)$...]

14.3 Main example

Let $G = U(2,2)$. Choose a quadratic imaginary field $F$, e.g. $\mathbb{Q}(i)$. Let $V = F^\oplus 4$ with the Hermetian form $Q(x, y, z, w) = x\bar{y} - z\bar{w}$. The group $G(2,2)$ is the symmetries of the Hermetian form $Q$. The maximal torus of $G$ has real rank two. There are two maximal parabolics.
The Klingon parabolic stabilizes the isotropic line. It has Levi subgroup isomorphic to $F^\times \times U(1,1)$.

The Siegel parabolic stabilizes the isotropic plane. This has Levi subgroup equal to $GL_2(F)$.

One will have $Y \subset \bar{Y} \supset \partial = \bar{Y} \setminus Y$. The boundary $\partial$ contains a $\partial_P$, which “looks like” a nil bundle over $Y_M$, where $M$ is a Levi for $P$.

We have a sequence

$$\cdots \longrightarrow H^i_c(Y_G) \longrightarrow H^i(Y_G) \longrightarrow H^i(\partial) \longrightarrow \cdots$$

This let’s reduce everything to $H^i(\partial)$, and even better $H^i(Y_P)$, or $H^{i+1}(Y_P)$. From this long exact sequence, we see that to attach Galois representations to systems of Hecke eigenvalues appearing in $H^i(\partial_M)$, it suffices to do so for systems of Hecke eigenvalues appearing in $H_c^i(Y_G)$.

We can pass to the completed cohomology, getting a similar commutative diagram, and obtain $\tilde{H}_c^i(\partial_P) = \tilde{H}_c^i(\partial_M)$. Using Hochschild-Serre, it suffices to consider $\tilde{H}_c^i(Y_G)$. For this, think of $\tilde{H}_c^i(Y_G)$ as étale cohomology of a perfectoid space (a Shimura variety “at infinite level”) and use a comparison theorem to compare with coherent cohomology. Ultimately, this reduces to classical modular forms on $U(2,2)$. Thanks to the work of many people, we already know how to associate Galois representations to holomorphic forms on $U(2,2)$. We can “chase these back” to attach Galois representations to cohomology classes of hyperbolic three-manifolds.

Of course, Scholze does this in far greater generality, i.e. for $U(n,n)$, or $GL_n$ of a totally real or CM field.
15 Future directions 1: formal $\mathbb{Q}_p$-vector spaces of slope $> 1$

As in my last talk, there will be many “open problems,” i.e. basic questions about perfectoid spaces whose answer is not known. Much of this is motivated by discussions with Fargues and Weinstein.

15.1 Crystals and Dieudonné modules

Let $k$ be a perfect field. Let $\mathcal{O}_{K_0} = W(k)$ be the ring of $p$-typical Witt vectors over $k$, and let $K_0 = \mathcal{O}_{K_0}[[t]]$. Let $G$ be a $p$-divisible group over $k$.

**Theorem 15.1.1.** There is an anti-equivalence of categories:

$\{p\text{-divisible groups over } k\}^{\circ} \simeq \{(M, F, V) : M \text{ is a finite free } \mathcal{O}_{K_0}\text{-module, } F : M \to M \text{ is } \sigma\text{-linear, } V : M \to M \text{ is } \sigma^{-1}\text{-linear and } VF = FV = p\}$

**Definition 15.1.2.** An isocrystal is a finite free $K_0$-vector space $N$ together with $\varphi : N \to N$, a $\sigma$-linear automorphism.

**Theorem 15.1.3** (Dieudonné-Manin). Assume $k$ is algebraically closed. Then any isocrystal $(N, \varphi)$ is isomorphic to $\bigoplus_i N_{\lambda_i}$, where $\lambda_i \in \mathbb{Q}$.

The $N_{\lambda_i}$ are “isocrystals of slope $\lambda_i$.” If $\lambda = \frac{s}{r}$ with $(s, r) = 1$ and $r > 0$, then as a vector space, $N_{\lambda} = K_0^{\oplus r}$ if $0 \leq s < r$. The Frobenius is the matrix

$$\varphi_{N_{\lambda}} = \begin{pmatrix}
p & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & p
\end{pmatrix}, \begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix}$$

where there $rps$ and $(r - s)1$s. The other cases are obtained by twisting.

If all the $\lambda_i \in [0, 1]$, then these come from formal $p$-divisible groups (which are unique up to isogeny).

**Question 15.1.4.** Are there objects that correspond to general isocrystals of slopes $\geq 0$?

15.2 The universal cover

Let $G$ be a $p$-divisible group. For convenience, assume $G$ is actually a formal group.

**Definition 15.2.1.** The universal cover of $G$ is the functor

$$\tilde{G} : k\text{-Alg} \to \text{Set}$$

$$\tilde{G}(R) = \lim_{\times p} G(R)$$
Proposition 15.2.2 (Fargues, Fontaine). Let \( d = \dim G \). Then \( \tilde{G} \) is represented by \( \text{Spf } k \left[ \frac{T_1}{p^{\infty}}, \ldots, \frac{T_d}{p^{\infty}} \right] \).

Consider the ring \( k \left[ \frac{T_1}{p^{\infty}}, \ldots, \frac{T_n}{p^{\infty}} \right] \) as an inverse limit of its quotients by ideals consisting of power series with degree \( \geq i \).

Definition 15.2.3. 1. A \( k \)-algebra \( R \) is semiperfect if \( \Phi : R \to R \) is surjective. Then \( R^\flat = \lim_{\leftarrow} \text{R} \) surjects to \( R \). Let \( I = \ker(R^\flat \to R) \).

2. A semiperfect ring \( R \) is balanced if \( \Phi(I) = I^p \subset R^\flat \).

Example 15.2.4. The ring \( k[\frac{T_1}{p^{\infty}}]/T \) is balanced. One has \( \Phi(I) = (T_1^p) \).

Example 15.2.5. The ring \( k[\frac{T_1}{p^{\infty}}, \frac{T_2}{p^{\infty}}]/(T_1, T_2) \) has \( \Phi(I) = (T_1^p, T_2^p) \), but \( I^p = (T_1^p, T_1^{-1}T_2, \ldots, T_2^p) \). So this ring is not balanced.

Note that \( \tilde{G} \) is determined by its values on balanced semiperfect rings.

Proposition 15.2.6 (Fontaine). If \( R \) is a semiperfect ring, then there is a universal \( p \)-adically complete PD-thickening \( A_{\text{cris}}(R) \to R \).

Proof. We briefly recall the construction of \( A_{\text{cris}}(R) \). It is the \( p \)-adic completion of the PD hull of the kernel of \( W(R^\flat) \to R \).

For example, \( A_{\text{cris}}(O_C/p) \) is the usual ring \( A_{\text{cris}} \). In general, the ring \( A_{\text{cris}}(R) \) may have lots of \( p \)-torsion.

Proposition 15.2.7. If \( R \) is balanced, then \( W(R^\flat) \to A_{\text{cris}}(R) \) is injective.

This is false in general.

Recall that \( B_{\text{cris}}^+(R) = A_{\text{cris}}(R)[\frac{1}{p}] \). We have a Frobenius morphism \( \varphi \) acting on both rings.

Theorem 15.2.8 (Scholze, Weinstein, rem. of Lam). Let \( G \) be a \( p \)-divisible group over \( k \) with universal cover \( \tilde{G} \) and isocrystal \( (N, \varphi) \). Then for any balanced semiperfect \( R \), one has \( \tilde{G}(R) = \text{hom}_{\varphi}(N, B_{\text{cris}}^+(R)) \).

Proof. We will define the map. Recall that \( \tilde{G}(R) = \text{hom}_R(\mathbb{Q}_p/\mathbb{Z}_p, G)[\frac{1}{p}] \). Thus \( \tilde{G}(R) \) is

\[
\text{hom}_{B_{\text{cris}}^+(R), \varphi}(N \otimes_{K_0} B_{\text{cris}}^+(R), B_{\text{cris}}^+(R)) = \text{hom}_{\varphi}(N, B_{\text{cris}}^+(R)).
\]

In some sense, this is the fully-faithfulness of the Dieudonné-module functor over \( R \).

Note that the right-hand-side in the theorem makes sense even if \( N \) does not come from some \( G \).

Definition 15.2.9. Let \( N \) be any \( k \)-isocrystal of slope \( \geq 0 \). Define \( \tilde{G}_N \) as a functor from balanced semiperfect \( k \)-algebras of slope \( > 0 \). to sets, by

\[
R \mapsto \text{hom}_{\varphi}(N, B_{\text{cris}}^+(R)).
\]
Question 15.2.10. Is \( \tilde{G}_N \) representable, i.e. \( \tilde{G}_N = \text{Spf}(R_N) \) for some inverse limit \( R_N \) of balanced semiperfect rings?

The Frobenius \( \Phi : R \to R \) induces a bijection \( \tilde{G}_N(R) \cong \tilde{G}_N(R) \). So such a \( R_N \) would be the perfection of some balanced semiperfect ring.

Let’s specialize to the first interesting case: \( N = N_2 \). So \( N = (K_0, \varphi = p^2) \). In this case we put \( \tilde{G}_2 = \tilde{G}_{N_2} \), which is \( R \to B^{\varphi=p}_\text{cris}(R) \). There is a good candidate for a functor representing this ring. We have a map \( \tilde{G}_1 \otimes \tilde{G}_1 \to \tilde{G}_2 \), which comes from the multiplication map

\[
B^{\varphi=p}_\text{cris}(R) \otimes B^{\varphi=p}_\text{cris}(R) \to B^{\varphi=p}_\text{cris}(R).
\]

Recall that \( \tilde{G}_1 = \tilde{G}_{N_2} \), so \( \tilde{G}_1 \times \tilde{G}_1 = \text{Spf}(k \left[ X^{1/p^\infty}, Y^{1/p^\infty} \right]) \).

Note that \( \mathbb{Q}_p^\times \) acts on \( \tilde{G}_1 \), via \( X \mapsto (1 + X)^\gamma - 1 \) for \( \gamma \in \mathbb{Q}_p^\times \). To make this precise, write \( \gamma = p^i\gamma_0 \) for \( \gamma_0 \in \mathbb{Z}_p^\times \). Then

\[
((1 + X)^{p^i})^{\gamma_0} - 1 = \sum_{n \geq 1} \left( \binom{\gamma_0}{n} \right) X^{p^i-n}.
\]

We get a map \( \text{Spf} \left( k \left[ X^{1/p^\infty}, Y^{1/p^\infty} \right] / (\mathbb{Z}/2 \times \mathbb{Q}_p^\times) \right) \to \tilde{G}_2 \)

If \( \tilde{G}_2 \) is representable, the representing object should be \( R_2^0 \), which is the ring of invariants

\[
R_2^0 := k \left[ X^{1/p^\infty}, Y^{1/p^\infty} \right] ^{\mathbb{Z}/2 \times \mathbb{Q}_p^\times}.
\]

I suspect that if \( \tilde{G}_2 \) is representable, then the representing object is this ring \( R_2^0 \).

If \( \tilde{G}_2 \) is represented by \( R_2 \), then there is a map \( R_2 \to R_2^0 \), so \( R_2^0 \) has to be quite big.

Question 15.2.11. Is \( R_2^0 \neq k \)?

If \( k \left[ X^{1/p^\infty}, Y^{1/p^\infty} \right] ^{\mathbb{Z}_p^\times} \neq k \), then \( R_2^0 \neq k \). One shows this by summing up over iterates of Frobenius. The group \( \mathbb{Z}_p^\times \) is (up to something finite) a topologically cyclic group (at least if \( p > 2 \)).

Let’s look at the action of \( 1 + p\mathbb{Z}_p \), which we can regard as the group of transformations of the form \( f(X) = X + a_1 X^p + a_2 X^{p^2} + \cdots \). Topologically, the latter group is generated by \( X + pX^p \). The task is to find \( F(X, Y) \in k \left[ X^{1/p^\infty}, Y^{1/p^\infty} \right] \) such that for \( f(X) = X^p \), \( f^{-1}(X) = X + X^p + X^{p^2} + \cdots \), one has

\[
F(X - X^p, Y + Y^p + Y^{p^2} + \cdots) = F(X, Y).
\]

Alternatively, we could look for \( F = F(X, Y) \) such that

\[
F(X + X^p, Y) = F(X, Y + Y^p).
\]
16 $p$-adic Hodge theory for rigid spaces 2

16.1 Hodge-Tate spectral sequence

Let $C$ be an algebraically closed complete extension of $\mathbb{Q}_p$. Let $X$ be a proper smooth rigid-analytic variety over $C$. Recall there is a Hodge-de Rham spectral sequence

$$E_1^{ij} = H^j(X, \Omega^i_X) \Rightarrow H^{i+j}_{dR}(X),$$

obtained from the Hodge filtration on the de Rham complex. One lets $\text{Fil}^k \Omega^\bullet_X$ come from the “truncated de Rham algebra” $\Omega^\bullet_X^{\geq k}$. For example, if $X$ is an honest scheme over $C$, then the Hodge-de Rham spectral sequence vanishes in sufficiently large degree. In general, one reduces by “spreading to discrete valuation rings.”

Theorem 16.1.1. There is a Hodge-Tate spectral sequence

$$E_2^{ij} = H^i(X, \Omega^j_X(-j)) \Rightarrow H^{i+j}_{\text{ét}}(X, \mathbb{Q}_p \otimes \mathbb{Q}_p).$$

Example 16.1.2. Let $X$ be a scheme over $C$. Then the Hodge-Tate sequence degenerates at $E_2$. One would expect that the Hodge-Tate sequence always degenerates on the second page.

We now prove Theorem 16.1.1. This is the descent spectral sequence for for $\nu : X_{\text{pro-ét}} \to X_{\text{ét}}$.

Lemma 16.1.3. There is a natural isomorphism

$$H^i_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \simeq H^i(X_{\text{pro-ét}}, \mathcal{O}_X).$$

Proof. The basic comparison from my last lecture yields an almost isomorphism

$$H^i_{\text{ét}}(X, \mathbb{Z}/p) \otimes_{\mathbb{Z}/p} \mathcal{O}_C \simeq H^i(X_{\text{ét}}, \mathcal{O}_X^+/p).$$

We pass by devissage to an almost isomorphism

$$H^i_{\text{ét}}(X, \mathbb{Z}/p^n) \otimes_{\mathbb{Z}/p^n} \mathcal{O}_C/p^n \simeq_a H^i(X_{\text{ét}}, \mathcal{O}_X^+/p).$$

We can pass to the inverse limit to obtain

$$H^i(X_{\text{pro-ét}}, \mathcal{O}_X^+) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \simeq_a H^i(X_{\text{pro-ét}}, \mathcal{O}_X^+).$$

Invert $p$ to obtain the comparison between étale and pro-étale cohomology.

Lemma 16.1.4. That there is a natural isomorphism

$$R^j\nu_* \mathcal{O}_X \simeq \Omega^j_{X_{\text{ét}}}(-j).$$
Proof. We claim that $\mathcal{E} = R^1\nu_*\widehat{\mathcal{O}}_X$ is a locally free $\mathcal{O}_{X,K}$-module, of rank $d = \dim X$, such that $\Lambda^j \mathcal{E} \simeq R^j\nu_*\widehat{\mathcal{O}}_X$ for all $j \geq 0$. Work locally and assume we have “good coordinates” witnessed by a map $X \to T$, where $T = \text{Spa}(C(T^{\pm 1}), \mathcal{O}_C(\cdots))$, and the map is a composite of rational maps and open embeddings. Let $\widetilde{T}$ correspond to passing to $T_{i/p}$. Then $\widetilde{T} \to T$ is a $\mathbb{Z}_p$-pro-covering. Let $\widetilde{X} = X \times_T \widetilde{T}$; this is a $\mathbb{Z}_p$-pro-covering of $X$. We have $H^i(X_{\text{pro-ét}}, \widehat{\mathcal{O}}_X) = H^i_{\text{cont}}(\mathbb{Z}_p, M)$, where $M = \mathcal{O}_{\widetilde{X}}(\widetilde{X})$. We can write $M$ explicitly as

$$M = \mathcal{O}_{\widetilde{X}}(\widetilde{X}) = \mathcal{O}_X(X) \otimes_{C(T_{i/p}^{\pm 1})} C(T_{i/p}^{\pm 1}).$$

We now compute

$$H^i_{\text{cont}}(\mathbb{Z}_p, M) = \mathcal{O}_X(X) \otimes H^i_{\text{cont}}(\mathbb{Z}_p, C(T_{i/p}^{\pm 1}))$$

$$= \mathcal{O}_X(X) \otimes H^i_{\text{cont}}(\mathbb{Z}_p, C(T_{i}^{\pm 1}))$$

$$\simeq \mathcal{O}_X(X) \otimes \Lambda^i C(T_{i}^{\pm 1}).$$

We conclude that $H^0(X_{\text{pro-ét}}, \widehat{\mathcal{O}}_X) \simeq \mathcal{O}_X(X)$ and $H^i(X_{\text{pro-ét}}, \widehat{\mathcal{O}}_X) \simeq \Lambda^i H^1(X_{\text{pro-ét}}, \widehat{\mathcal{O}}_X)$ for $i \geq 1$.

\section{16.2 Relative period rings}

\textbf{Lemma 16.2.1.} There is a natural isomorphism $\mathcal{E} \simeq \Omega^1_{X,K}(-1)$.

\textit{Proof.} Why should this be true? Assume $X$ is defined over $\text{Spa}(K, \mathcal{O}_K)$, where $K$ is a complete discrete-valuation field with perfect residue field. Then we have a “Poincaré lemma.” There is a “relative linearization of this complex.” There is a resolution of $\text{Spec } X$ by acyclic crystals, and we can evaluate this resolution of $\mathcal{A}^+_{\text{cris}}$.

Here there is an exact sequence $0 \to \mathcal{E} \to \mathcal{L}(\mathcal{F}) \to \mathcal{L}(\mathcal{F} \otimes \Omega^1) \to \cdots$. This yields

$$0 \to \mathcal{E}(\mathcal{A}_{\text{cris}}) \to \mathcal{L}(\mathcal{F})(\mathcal{A}_{\text{cris}}) \to \mathcal{L}(\mathcal{F} \otimes \Omega^1)(\mathcal{A}_{\text{cris}}) \to \cdots$$

We want a Faltings extension from the Poincaré lemma. Filter the Poincaré lemma by $(\ker \theta)^i$, and look at the $i$-th graded piece. In $\mathcal{O}_X$, $\ker \theta = (t, u_1, \ldots, u_d)$. Tere is an exact sequence

$$0 \to \mathcal{O}_X(1) \to \text{gr}_k^i \mathcal{O}_X \to \mathcal{O}_X \otimes \Omega^1_X \to 0$$

Applying $R\nu_*$, and we get

$$0 \to \nu_* \mathcal{O}_X(1) \to \nu_* \text{gr}_k^i \mathcal{O}_X \to \nu_* (\mathcal{O}_X \otimes \Omega^1_X) \xrightarrow{\partial} R^1 \nu_* \mathcal{O}_X(1) \to R^1 \nu_* \text{gr}_k^i \mathcal{O}_X$$
We want \( \partial \) to be an isomorphism, so we prove that
\[
\nu_* \text{gr}_F^1 \mathcal{O}_\text{dR}^+ = R^1 \nu_* \text{gr}_F^1 \mathcal{O}_\text{dR}^+ = 0.
\]
It is known that \( R^k \nu_* \text{gr}_F^1 \mathcal{O}_\text{dR} = 0 \) for \( k \geq 0 \).

**Example 16.2.2.** Let \( A \) be an abelian variety over \( C \). Then the Hodge-de Rham spectral sequence is
\[
0 \to H^0(A, \Omega^1_A) \to H^1_\text{dR}(A) \to H^1(A, \mathcal{O}_A) \to 0.
\]
The Hodge-Tate spectral sequence is
\[
0 \to H^1(A, \mathcal{O}_A) \to H^1_{\text{et}}(A, \mathbb{Z}_p) \otimes \mathbb{Z}_p \to H^0(A, \Omega^1_A)(-1) \to 0. \tag{HT1}
\]
Assume \( A \) has good reduction, and denote also by \( A \) a model for \( A \) over \( \mathcal{O}_C \). Let \( G = A[p^\infty] \) be the associated \( p \)-divisible group.

**Theorem 16.2.3** (Faltings, Fargues). The complex of finite free \( \mathcal{O}_C \)-modules
\[
0 \longrightarrow (\text{Lie } G)(1) \longrightarrow TG \otimes_{\mathbb{Z}_p} \mathcal{O}_C \alpha \longrightarrow (\text{Lie } G^*)^* \longrightarrow 0 \tag{HT2}
\]
has cohomology annihilated by \( p^{1/(p-1)} \).

Here, \( TG \) is the Tate module of \( G \). If \( G \) is defined over some field \( L \), then \( \alpha_G \) can be defined over the field generated by \( L \) and the torsion-points of \( TG \). One defines \( \alpha_G \) as follows. Let \( \alpha \in TG = \lim G[p^n](\mathcal{O}_C) = \text{hom}_{\mathcal{O}_C}(\mathbb{Q}_p/\mathbb{Z}_p, G) \). Thus \( \alpha : \mathbb{Q}_p/\mathbb{Z}_p \to G \). We can dualize to get \( \alpha^* : G^* \to \mu_{p^\infty} \). Take Lie algebras to get \( \text{Lie}(G^*) \to \text{Lie}(\mu_{p^\infty}) = \mathcal{O}_C \).

**Theorem 16.2.4.** The sequences (HT1) and (HT2) are dual to each other.

In other words, we claim that the following sequences are dual:
\[
0 \longrightarrow \text{Lie } A^* \otimes_{\mathcal{O}_C} C \longrightarrow H^1_{\text{et}}(A_C, \mathbb{Z}_p) \otimes \mathbb{Z}_p \longrightarrow (\text{Lie } A)^* \otimes_{\mathcal{O}_C} C(-1) \longrightarrow 0
\]
\[
0 \longrightarrow \text{Lie } A \otimes_{\mathcal{O}_C} C(1) \longrightarrow H^1_{\text{et}}(A_C, \mathbb{Z}_p) \otimes C(1) \longrightarrow (\text{Lie } A^*)^* \otimes_{\mathcal{O}_C} C \longrightarrow 0
\]
One sees this using the Weil pairing.
17 Shimura varieties and perfectoid spaces

Fix a genus \( g \geq 1 \). Also fix a prime-to-\( p \) level \( K^p \subset \text{GSp}_{2g}(A_f^p) \). There are spaces \( X_{\Gamma(p^m)} \), \( X_{\Gamma_0(p^m)} \) which are moduli spaces of principally polarized abelian varieties of dimension \( g \), level \( \Gamma \). More precisely, these are minimal compactifications of the moduli spaces. One can imagine \( g = 1 \), in which case we are looking at modular curves.

17.1 Moduli spaces of abelian varieties

Consider \((K, K^+), K^c_p\), a complete non-archimedean field. Define 
\[
X_{\Gamma(p^\infty)}(K, K^+) = \varprojlim X_{\Gamma(p^n)}(K).
\]
If \( x \) is a “\((K, K^+)\)-valued point of \( X_{\Gamma(p^\infty)} \),” not in the boundary, then we get an abelian variety \( A \) over \( K \), together with an isomorphism \( A[p^\infty](K) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{2g} \). In particular, 
\[
\text{Lie } A(1) \subset T_p A \simeq \mathbb{Z}_p^{2g} \otimes K
\]
is an isotropic \( g \)-dimensional subspace of \( K^{2g} \). Let \( F(K) \) be the Grassmannian of such subspaces.

Summing up, we get a map \( X_{\Gamma(p^\infty)} \circ \to F(K) \).

Theorem 17.1.1. There is a perfectoid space \( X_{\Gamma(p^\infty)} \), defined over \( \mathbb{Q}^{cyc}_p \), such that 
\[
X_{\Gamma(p^\infty)} \sim \varprojlim X_{\Gamma(p^n)} \to F
\]
is perfectoid.

In fact, the theorem says that \( \varprojlim X_{\Gamma(p^n)} \) is perfectoid. Moreover, the Hodge-Tate period map \( \pi_{HT} \) is \( \text{GSp}_{2g}(\mathbb{Q}_p) \)-equivariant.

Example 17.1.2. Suppose \( g = 1 \). Then \( |X_{\Gamma(p^\infty)}| = X_{\Gamma(p^\infty)}^{\text{ord}} \cup X_{\Gamma(p^\infty)}^{\text{ss}} \). The variety \( F \) is the projective space line, so \( \pi_{HT} \) is a map to \( \mathbb{P}^1 \). It maps \( X_{\Gamma(p^\infty)} \) to \( \mathbb{P}^1(\mathbb{Q}_p) \), and maps \( X_{\Gamma(p^\infty)}^{\text{ss}} = \text{LT}_{\infty} \simeq \Omega_{\infty} \) to the Drinfeld upper half-plane \( \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{Q}_p) \).

17.2 Sketch of proof of main theorem

We’ll mostly concentrate on the fact that the “infinite level” Shimura variety is a perfectoid space. The idea is to prove perfectoidness on an open part of \( X_{\Gamma(p^\infty)} \), and then use the action of \( \text{GSp}_{2g}(\mathbb{Q}_p) \). After “moving things around” by the action of the group, we’ll get everything.

For any \( 0 \leq \varepsilon < 1 \) and \( \Gamma = \Gamma_0(p^n) \) or \( \Gamma = \Gamma_1(p^n) \) we’ll define \( X_\Gamma(0) \subset X_\varepsilon(0) \subset X_\Gamma(1) \), which is the locus where the Hasse invariant \( H \) satisfies \( |H| \geq p^{-\varepsilon} \). For \( \Gamma = \Gamma(1) \), we’ll write \( X(0) \subset X(\varepsilon) \subset X(1) \).

Proposition 17.2.1. Let \( 0 \leq \varepsilon < \frac{1}{2} \).

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1. For \( m \geq 1 \), the abelian variety \( A(p^{-m}\varepsilon) \to X(p^{-m}\varepsilon) \) admits a “canonical subgroup” \( C_m \subset A(p^{-m}\varepsilon)[p^m] \).

2. The operation \( A \mapsto A/C_1 \) induces a map \( \tilde{F} : X(p^{-m}\varepsilon) \to X(p^{-m+1}\varepsilon) \), which reduces to Frobenius modulo \( p^{1-\varepsilon} \) (on the natural integral models).

3. The operation \( (A, C_m) \mapsto (A/C_m, A[p^m]/C_m) \) induces a morphism \( X(p^{-m}\varepsilon) \to X_{\Gamma_0(p^m)} \).

This induces an isomorphism onto a local subset \( X_{\Gamma_0(p^m)}(\varepsilon)_a \subset X_{\Gamma_0(p^m)}(\varepsilon) \). (The \( (-)_a \) stands for “anti-canonical.”) This is summarized in the following diagram:

\[
\begin{array}{ccc}
X(p^{-m-1}\varepsilon) & \xrightarrow{\tilde{F}} & X_{\Gamma_0(p^{m+1})} \\
| & | & | \\
X(p^{-m}\varepsilon) & \xrightarrow{\tilde{F}} & X_{\Gamma_0(p^m)} \\
| & | & | \\
X_{\Gamma_0(p)}(\varepsilon)_a & \subseteq & X_{\Gamma_0(p)} \\
\end{array}
\]

The above diagram is commutative and Cartesian.

Define

\[
X_{\Gamma_0(p^\infty)}(\varepsilon)_a = \lim_{\varepsilon \to \infty} X_{\Gamma_0(p^m)}(\varepsilon)_a = \lim_{\varepsilon \to \infty} X(p^{-m}\varepsilon)
\]

From part 2 of the proposition, we see that this is perfectoid. There is a map \( X_{\Gamma(p^\infty)} \to X_{\Gamma_0(p^\infty)} \) taking \( (A, \alpha : T_p A \simeq \mathbb{Z}_p^2) \) to \( (A, (\alpha^{-1}(\mathbb{Z}_p^2 \oplus 0))) \).

Consider \( X_{\Gamma(p^\infty)}(\varepsilon)_a \) to be the preimage of \( X_{\Gamma_0(p^m)}(\varepsilon)_a \). Put

\[
X_{\Gamma(p^\infty)}(\varepsilon)_a = \lim_{\varepsilon \to \infty} X_{\Gamma(p^m)}(\varepsilon)_a.
\]

We claim that this inverse limit is perfectoid.

The locus where \( X_{\Gamma(p^\infty)} \) is stable under the action of \( \text{GSp}_{2g}(\mathbb{Q}_p) \). I claim that

\[
X_{\Gamma(p^\infty)}(\varepsilon) = \text{GSp}_{2g}(\mathbb{Z}_p) \cdot X_{\Gamma(p^\infty)}(\varepsilon)_a,
\]

and the space on the left is perfectoid.

**Lemma 17.2.2.**

1. \( \pi_{HT}^{-1}(F(\mathbb{Q}_p)) = X_{\Gamma(p^\infty)}(0) \).

2. There exists an open neighborhood \( F \supset U \supset F(\mathbb{Q}_p) \), such that \( \pi_{HT}(U) \subset X_{\Gamma(p^\infty)}(\varepsilon) \).

3. \( \text{GSp}_{2g}(\mathbb{Q}_p) \cdot X_{\Gamma(p^\infty)}(\varepsilon) \supset \pi_{HT}^{-1}(\text{GSp}_{2g}(\mathbb{Q}_p) \cdot U) \).

**Lemma 17.2.3.** If \( U \subset F \) is open and contains \( F(\mathbb{Q}_p) \) and is stable under \( \text{GSp}_{2g}(\mathbb{Q}_p) \), then \( U = F \).

**Proof.** We’ll give an argument if \( g = 1 \). Let \( x = \mathbb{Q}_p \oplus 0 \subset \mathbb{P}^1(\mathbb{Q}_p) \). If \( B \ni x \) is some ball in \( \mathbb{P}^1 \setminus \{\infty\} \). Then \( \left( \frac{1}{p} \right)^n \cdot B \subset U \) for \( n \gg 0 \), whence \( B \subset \left( \frac{1}{p} \right)^{-n} \cdot U = U \). □
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The point is to use the result of the last lecture, along with the Hodge-Tate period map, to compute the completed cohomology and its’ Hecke action.

Let $(G, X) \subset (\text{GSp}^\pm)$ be a Shimura datum of Hodge type. Let $K^p \subset G(K^p)$ be a level, and let $K_p \subset G(\mathbb{Q}_p)$. Write $Y_{K_p, K^p}$ for the corresponding Shimura variety, and $X_{K_p, K^p}$ for a minimal compactification. We think of everything as living over $\mathbb{C}_p$.

18.1 A theorem on Hecke actions

Recall that we defined completed cohomology as

$$\tilde{H}^i_{c,K^p}(\mathbb{Z}/p^n) = \lim_{K_p \to 0} H^i_c(Y_{K_p, K^p}, \mathbb{Z}/p^n).$$

Let $\mathbb{T} = \mathbb{Z}[G^P \backslash G(K^p)/K^p]$ be the corresponding Hecke algebra. This acts on $\tilde{H}^i_{c,K^p}(\mathbb{Z}/p^n)$.

There is a “standard” line bundle $\omega$ on $X_{K_p, K^p}$, coming from $\Omega^p$ of the universal family of abelian varieties. Let $I$ be the ideal sheaf of $X \smallsetminus Y$. The algebra $\mathbb{T}$ acts on a space of “modular forms” $H^0(X_{K_p, K^p}, \omega^k \otimes I)$.

**Theorem 18.1.1.** Let $\mathbb{T}_\text{cl}$ be the minimal quotient of $\mathbb{T}$ such that the action of $\mathbb{T}$ on $H^0(X_{K_p, K^p}, \omega^k \otimes I)$ factors through $\mathbb{T}_\text{cl}$ for all $k, K_p$. Then the action of $\mathbb{T}$ on $\tilde{H}^i_{c,K^p}(\mathbb{Z}/p^n)$ factors through $\mathbb{T}_\text{cl}$.

Let $X = \lim_{K_p \to 0} X_{K_p, K^p}$. This as denoted by $X_{G, \Gamma(p^\infty)}$ in the last talk, and we know it is a perfectoid space. Let $I^+ = \mathfrak{O}^+ \cap I$.

**Proposition 18.1.2.** There is an almost isomorphism

$$\tilde{H}^i_{c,K^p}(\mathbb{Z}/p^n) \otimes \mathcal{O}_{\mathbb{C}_p}/p^2 \simeq_a H^i(X, I^+/p^n I^+).$$

Write $j : Y \hookrightarrow X$ for the inclusion. We know that $H_{c,K^p}$ comes from $j^!\mathbb{Z}/p^n$. In other words, $H_i = H_{K^p}(j^!\mathbb{Z}/p^n)$. In this case, one has “one the nose” that $I^+/p^n = j_!\mathfrak{O}^+/p^n$. Near the boundary, one has $I^+ \to \mathfrak{O}^+ \to \mathfrak{O}_{\mathfrak{O}}^+$. To prove the theorem, we need to compute $H^i(X, I^+/p^n)$. For this we use the Hodge-Tate period map $\pi_{HT}$, coupled with the fact that $X$ is perfectoid.

We have a diagram

$$\begin{array}{ccc}
X_{\text{GSp}} & \xrightarrow{\pi_{HT}} & F^\bullet \\
\pi_{HT} \downarrow & & \downarrow \pi_{HT} \\
X & \xrightarrow{\pi_{HT}} & F^\bullet \text{(2g)}^{-1}
\end{array}$$

where $D \subset \mathbb{Z}^{2g} \otimes K$ is mapped to $\bigwedge^g D \subset \bigwedge^g (K^{2g})$. The pullback of $\mathfrak{O}(1)$ is $\omega$. The embedding from $F$ into projective space is via Plücker coordinates or something similar.

For $J \subset \{1, \ldots, 2g\}$ with $|J| = g$, let $F_J \subset F$ be the locus where $|S_J| \geq |S_{J'}|$ for all $J'$. For example,

$$\pi_{HT}^{-1}(F_{\{g+1, \ldots, 2g\}}(\mathbb{Q}_p)) = X_{\Gamma(p^\infty)}(0,g).$$

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Lemma 18.1.3. All $\pi^{-1}_{HT}(F_J)$ are affinoid perfectoid.

Proof. It is enough to consider $J = \{g + 1, \ldots, 2g\}$. Then $\gamma$ is a diagonal matrix with $g$ $p$-s, and then 1s along the diagonal. Consider $\gamma^n \cdot F_J \subseteq F_J$. These are rational subsets. If $n \gg 0$, then $\pi^{-1}_{HT}(\gamma^n \cdot F_J) \subseteq X_{\Gamma(p^n)}(\varepsilon)_a$, which is an affinoid.

[... not enough on board, did not follow...]

By the lemma, we can compute $H^i(X, I^+/p^n)$ using $\{\pi^{-1}_{HT}(F_J)\}_{J}$. Order these open subsets as $V_1, \ldots, V_N$, where $N = \binom{2g}{g}$. Let $J_2 \subset \{1, \ldots, N\}$, and put $V_{J_2} = \bigcap_{i \in J_2} V_i$. The $V_i$ are affinoid perfectoid. Then $H^i(V_{J_2}, I^+/p^n) = 0$ for all $i > 0$.

It is enough to show that the action of $T$ on $H^0(V_{J_2}, I^+/p^n)$ factors through $T_{cl}$. In fact, the $V_{J_2}$ come from $V_{J_2,K_p} \subset X_{K_p,K_p}$ for some $K_p$ small enough. The $V_{J_2,K_p}$ are affinoids, and bounded functions give a "natural" integral model $V_{J_2,K_p}^\circ$. One can "glue these" into a formal scheme $X_{K_p,K_p}^\circ$, equipped with an ample line bundle $\omega_{int}$ extending $\omega$.

Finally,

$$H^0(V_{J_2}, I^+/p^n) = \lim_{K_p} H^0(V_{J_2,K_p}^\circ, I^+/p)$$

$$= \lim_{K_p} \lim_{S_{J_2}} H^0(X_{K_p,K_p}^\circ, (\omega_{int})^\otimes k[J_{2}] \otimes J/p^n)$$

where $J$ is the ideal sheaf of the boundary on $X_{K_p,K_p}^\circ$. \[\square\]

[...to vague to follow + battery ran out...]

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References


