Goal: Construct Θ-stratifications
\[ X = X^{ss} \cup U S_a \]
along with good moduli space \( q: X^{ss} \rightarrow M \)

Keywords from first lecture:
- Filtrations \( F: \Theta \rightarrow X \)
- Numerical invariant \( \mu(f) \)
- Stability, HN problem

Thm A (HL):
If \( X \) is Θ-reductive, HN problem has a solution, and only finitely many HN types in bounded family \( \) defines Θ-stratification

Thm B (Alper-HL-Heinloth):
If \( X \) is bounded, it has a GMS iff
1) \( X \) is Θ-reductive, 2) unpunctured inertia, 3) closed points have reductive autom. groups.

Today: Discuss this theorem and applications to moduli of sheaves on a K3 surface

What is a Θ-reductive stack?
Def: A stack is Θ-reductive if...
- For any family over DVR \( \text{Spec}(R) \rightarrow X \), any filtration of the generic point extends to a filtration of the family

Ex 1: \( Y \) projective scheme, \( X = \text{Coh}(Y) \)
- Amounts to compactness of flag scheme
- Fails for \( \text{Bun}(Y) \)

Ex 2: More generally, proof can be adapted to \( X = \{ \text{objects in } A^3 \}, \) an abelian category

Ex 3: Quotient stacks \( \text{Spec}(A)/G \) are reductive

Prop: if \( X \) is Θ-reductive and a numerical invariant \( \mu \) defines a Θ-stratification, then \( X^{ss} \) is Θ-reductive.
Def: given a family over a DVR, we say that another map is a modification if the maps are isomorphic over the generic point \( \text{Spec}(K) \):

\[
\text{Spec}(R) \to X
\]

Ex: family of bundles on \( C \times \text{Spec}(R) \), \( E_0 = \) special fiber, \( F \subset E_0 \) sub-bundle, \( E' = \ker(E \to i_*(E_0/F)) \) — new bundle, elementary modification

Def: \( X \) has unpunctured inertia if for any family over a DVR, one can find an elementary modification such that

1) any connected component of \( \text{Aut}(\text{generic fiber}) \) specializes to special fiber, or
2) at finite-order generic automorph.

Amplifications

If \( q: X \to M \) is a good moduli space, then

\[ \longrightarrow M \text{ is separated if any modification over a DVR can be factored into sequence of elementary modifications} \]

\[ \longrightarrow M \text{ is proper if it is separated and } X \text{ satisfies existence part of valu. crit.} \]

Thm (semistable reduction):
Given a \( \Theta \)-stratification of \( X \), any family \( \text{Spec}(R) \to X \) with semistable generic point is related by elementary modification to a semistable family

Consequence: Can specify conditions on \( X \) s.t. the good moduli space of \( X^{ss} \) is proper
Slope semistability

Set up: $A^{0}D^{b}(Y) \text{ heart of bounded } t\text{-structure}$

$\sigma: v: K_{0}(Y) \rightarrow \Lambda = K_{0}^{num}(Y)$

$\mathcal{L}: \Lambda \rightarrow \mathbb{C}, \text{ -deg}(v) + \text{tr}(v) := \mathcal{L}(v)$

Pre-stability condition: all $E \in A$ have HN filtrations

Hypothesis:

$X_{v}(B) := \left\{ E \in D^{b}(Y \times B) \text{ s.t. } \right\}$

$E_{b} \in A \text{ for all } b \in B$ is an algebraic stack

Ex 1: Slope semistability in Coh$(Y)$

Ex 2: Any Bridgeland stability condition with $A$ noetherian

Rem: can work in more general categories, and can define $X$ directly from $A$

Moduli spaces

Central charge defines a line bundle: on $X_{v}$

Write $\mathcal{L}(E) = \chi(E \otimes \omega_{x}), \omega_{x} \in K_{0}^{num}(Y) \otimes \mathbb{C}$

\[ \begin{array}{c}
\mathbb{X} \times Y \\
\downarrow
\end{array} \]

\[
\begin{array}{c}
\mathcal{L} \circ \text{ch}_{1}(\Phi E_{\text{univ}}^{Y \rightarrow x}(\text{Im}(\frac{-\omega_{x}}{\mathcal{L}(E)}))) \\
\text{b} \circ \text{ch}_{2}(\Phi E_{\text{univ}}^{Y \rightarrow x}(\text{Im}(\omega_{x}))) \\
\end{array}
\]

Consequences of main theorems: if $X_{v}^{ss}$ bounded

$\forall v \in \Lambda$, then

$\rightarrow X$ has $\Theta$-stratification by HN type

$\rightarrow X_{v}^{ss}$ has proper good moduli space

Main example: From now on, we consider only Bridgeland stability on a smooth surface

$\Rightarrow$ we can study Donaldson invariants
We will always work with K-theoretic invariants:

\[ F \in K^0(X) \]  

where \( E \in K^0(Y) \) is obtained from \( Y \) by simple operations

\[ R_{q_*} : D^b(X^{ss}) \to D^b(M^{ss}_v) \] is well-defined by properties of GMS

Definition

\[ \mathcal{I}_v^\sigma(F) = \chi(M^{\sigma-ss}_v, R_{q_*}(F)) = \chi(X^{\sigma-ss}_v, F) \]

Question: how do \( X^{\sigma-ss}_v \) and \( I_v^\sigma(E) \) depend on stability condition \( \sigma \)? For nice results, we need to regard \( X^{\sigma-ss}_v \) as a derived stack.

- Algebraic geometry built commutative DGA's  
  \[ A \left[ e_1, \ldots, e_r ; d e_i = a_i \in A \right] \]
  \[ I_v^\sigma(F) = \text{integral over derived stack} \]

Correct Donaldson invariants of surfaces

A simple analogy for derived algebraic geometry:

\[ \text{reduced rings } \to \text{ rings } \to \text{ CDGA's} \]

On affine objects:

\[ H_0(A)_\text{red} \leftarrow H_0(A) \leftarrow A_0 \]

Analogous picture for derived schemes / stacks:

\[ \mathfrak{X}^{cl} \hookrightarrow \mathfrak{X}^{rig}, \text{ same underlying points} \]

Virtual structure sheaf: Note \( \bigoplus H_i(A_\bullet) \) is a coherent \( H_0(A_\bullet) \)-module

\[ \mathcal{O}_{\mathfrak{X}^{vir}} := \bigoplus H_i(A_\bullet)[i] e_i Y^{(i)}(x) \]

Classical shadow of derived world:

Given \( F \) on derived stack \( \mathfrak{X}^{rig} \),

\[ \chi(\mathfrak{X}^{rig}, F) = \chi(\mathfrak{X}^{rig}, F^{rig} \otimes_* \mathcal{O}_{\mathfrak{X}^{rig}}) = \chi(\mathfrak{X}^{cl}, \mathcal{L}^*(F) \otimes \mathcal{O}_{\mathfrak{X}^{cl}}) \]

Lecture 2 (collapsed) Page 5
Wall crossing

Situation: \( \text{Fix } \nu \in K_0^{\text{num}}(S) \)
- \( \sigma \) varies in a complex manifold \( \text{Stab}^*(s) \)
- in complement of real codim 1 walls, \( \chi^{s-ss}_\nu \) is constant
- let \( \sigma_0 \in \text{wall}, \sigma_1 \in \text{different chambers}, \chi^{s-ss}_\nu \) has GMS, and \( \chi^{s-ss}_\nu \rightarrow \chi^{s-ss}_{\sigma_1} \)
in some cases, as in last lecture

\[ \chi^{s-ss}_\nu = \chi^{s-ss}_{\sigma_1} \cup \bigcup_{\sigma \in \text{Stab}^*(s)} \chi_{\sigma} \bigcup_{\sigma \in \text{Stab}^*(s)} \chi_{\sigma} \cup \bigcup_{\sigma \in \text{Stab}^*(s)} \chi_{\sigma} \]

Hypothesis: \( \forall E \in A, L^{\text{vir}}_{x,E} = \text{RHom}(E, E(1))^* \)
has no cohomology in \( \deg \leq -1 \), i.e.,
\( \text{Hom}(E, E(1)) = 0 \)
for \( \nu \leq 2 \)

Rem: Holds automatically when \( A = \text{Cod}(S) \)
or when \( S \) is K3

Wall crossing formula

Thm: Under the previous hypotheses, we have
\[
I^0_\nu(S) - I^0_\mu(S) = \sum_{\alpha} \chi(\chi^{s-ss}_\nu \times \ldots \times \chi^{s-ss}_\nu, \text{centers of strata})
\]

Decompose \( L^{\text{vir}}_{x,E} \mid_{\text{center}} = L^+ \oplus L^0 \oplus L^- \)
\( E_x = \text{Sym}(L^+) \otimes \text{Sym}(L^-) \otimes \det(L^+) \otimes \det(L^-) \)

Proof idea:
Local cohomology uses derived AG, and
modular interpretation of the strata

Uses:
1) compute \( I^0_\nu(S) - I^0_\mu(S) \)
2) come up with explicit formulas
   For \( I^0_\nu \) by wall crossing to where
\( \chi^{s-ss} = \emptyset \)

Nagging question: Combinatorial structure?
Birational geometry -- K3 case

If $v$ is primitive and $\sigma$ generic, then

$$M^{\sigma-ss}_v = \text{smooth projective hyperkähler}$$

Restrict to class of CY manifolds birationally equivalent to $M^{\sigma-ss}_v$

Thm (Bayer-Macri): Any two manifolds in this class can be connected by a sequence of birational modifications of the form:

$$M^{\sigma_i-ss}_v \rightarrow M^{\sigma_i-ss}_v$$

For some (twisted) K3 surface $S$

We now have a diagram:

Local models for flops

Base change the picture:

$$\begin{array}{ccc}
\mathbb{E}^{\sigma-ss}_v & \leftarrow & \text{Spec}(A)/\Gamma_l \\
\downarrow & & \downarrow \\
M^{\sigma-ss}_v & \leftarrow & \text{Spec}(A^G) \\
\end{array}$$

One can show, using self-duality $L_x \cong L^{*}_{\infty}$

Thm: $\text{Spec}(A)$ is zero fiber of a "weak" algebraic moment map $\mu: \text{Spec}(B) \rightarrow \mathfrak{g}^*$ for a smooth affine $G$-scheme.

Application: we recently used this to prove that any two smooth projective CY manifolds in birational class of $M^{\sigma-ss}_v$ have equivalent derived categories of coherent sheaves.