When $X$ is a projective variety, $M(X)$ often has analogs in algebraic geometry. 

$\Rightarrow$ easier, thanks to language of stacks

First issue:

some objects are parameterized by continuous parameters

Ex: moduli of curves as an algebraic stack

= functor of points

For any scheme $B$, $M_g(B)$ := \{ smooth families of genus $g$ curves over $B$ \}

An algebraic stack which is "Deligne-Mumford" (DM)

$\Rightarrow$ finite automorphism groups

Why study moduli problems? Two reasons...

A. Classify geometric objects
   leads to deeper understanding of the structure of these objects

Ex: compact Lie groups vs. theory of root data

B. Explosion of interest in moduli since 80's

New invariants in differential geometry:
- pseudoholomorphic curves, gauge theory

Idea: Given a manifold $X$...

Associate a moduli problem $M(X)$, then "count" $\# M(X)$, objects up to isomorphism.
Gold standard:

**Thm (Keel-Mori):** Any separated DM stack $\mathcal{M}$ has a coarse moduli space $\mathcal{M} \to X$

Ignore: space vs. scheme vs. quasi-projective scheme

This suggests a general approach:

1) Identify a functor of points

2) Show that it is an algebraic stack

   e.g. Artin's criteria

3) Check that it is separated and DM, and apply Keel-Mori theorem

Problem (for either goals A or B): Many stacks of interest are not DM, so (3) fails

Today: A version of this program which works for general algebraic stacks

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Gold standard:

Fix smooth genus $g$ curve, $C$

$$\operatorname{Bun}_{r,d}(B) := \begin{cases} 
\text{vector bundles on } C \times B \\
\text{deg}=d, \text{rank}=r \text{ on fibers}
\end{cases}$$

This is an algebraic stack

can discuss line bundles, sheaves, cohomology, open/closed substacks

Ex: a line bundle is a natural assignment

$$(\text{maps } B \to \operatorname{Bun}_{r,d}) \to (\text{line bundles on } B)$$

But doesn't help much with goal A, classification:

$\operatorname{Bun}_{r,d}$ is unbounded, non-separated

(look at degenerates of $\mathcal{O}_C(-n) \oplus \mathcal{O}_C(n)$)
More interesting: sheaves on a surface \( S \)

\[
\mathcal{E}_V = \left\{ \text{coherent sheaves on } S \text{ with numerical } K\text{-theory class } v \right\}
\]

Exact same structure, but HN stratification depends on an ample class \( H \in \text{NS}(S)_{\mathbb{R}} \):

\[
\mathcal{X}_V = \mathcal{X}_V^{H=\text{ss}} \cup \bigcup \mathcal{S}_\alpha \quad \alpha = \left\{ (v_1, \ldots, v_t) \in K^{\text{num}}(S) \right\}
\]

Donaldson invariants of \( S \) arise as "integrals"

\[
\int_{H=\text{ss}} \left( \text{tautological } \right) \quad \int_{M^V} \left( \text{cohomology class} \right)
\]

Q: How do they depend on \( H \)?

Ex: Bridgeland semistable objects in \( \mathcal{A} \subset D^b(S) \)

New techniques needed
Good moduli spaces:
\( \mathcal{X} \) is a finite type stack, affine diagonal

**Def:** a good moduli space (GMS) is a map to a space 
\[ q : \mathcal{X} \to M \]
s.t.
1. \( q^* : \text{Qcoh}(\mathcal{X}) \to \text{Qcoh}(M) \) is exact
2. \( q^* (\Theta) = M \)

**Properties:**
1. Fibers of \( q \) = "S-equivalence" classes
2. Universal for maps to spaces (categorical quotient)
3. Example: \( X^{ss}/G \to X^{ss}/G \), reductive GIT
4. If \( M \) is proper, \( \dim H^*(\mathcal{X},\mathcal{E}) < \infty \)
5. Alper-Hall-Rydh '16: \( \mathcal{X} \) is locally over \( M \), looks like \( \text{Spec}(\mathcal{A})/G \to \text{Spec}(\mathcal{A}/G) \)

Solving moduli problems:

**Stability**
\( \mathcal{X} = \text{stack of coherent sheaves on } S \)

**Rees correspondence:**
\[ \begin{align*}
& (\mathbb{Q} \text{-weighted filtrations } \cdots \mathcal{E}_w \mathcal{E}_{w_1} \cdots ) \\
\Downarrow \\
& (C^* \text{-equiv. coherent sheaves on } S \times C^1) \\
\Downarrow \\
& (\text{maps of stacks } \Theta := C^* \to \mathcal{X})
\end{align*} \]

Give an intrinsic formulation of the HN filtration:

\[ \text{a canonical map } F : \Theta \to \mathcal{X} \]

How to find it? Use numerical invariant.

Given \( F : \Theta \to \mathcal{X} \), define: (assuming \( \text{deg}(v) = 0 \))
\[ \mu(F) = \frac{-\sum w \text{deg}(\mathcal{E}_w/\mathcal{E}_{w_1})}{\sum w^2 \text{rank}(\mathcal{E}_w/\mathcal{E}_{w_1})} \]
Main theorem of GIT:
Let $q : Y \to \Theta$ be a good moduli space, and fix a form $b \in H^4(Y)$.

Thm (HL, Hoskins, Zamora): Among all maps $f : \Theta \to X$, there one which maximizes $\mu(f)$, unique up to ramified cover $\Theta \to \Theta$.

Now let $X$ be arbitrary, and fix $x \in H^2(X)$ and $b \in H^4(Y)$.
1) $p \in X$ is semistable if $f^*(x) \leq 0$ in $H^2(\Theta) \cong \mathbb{Z}$ for all filtrations of $p$
2) [HN problem] For unstable $p \in X$, find $f$ which maximizes
$$\mu(f) = \frac{f^*(x)}{\sqrt{\det b}}$$

Might lead to a $\Theta$-stratification:
$$X = X^{ss} \cup U \Sigma,$$
where $\Sigma$ parameterize filtered objects $\Theta \to X$
$\Sigma^{ss}$ parameterize graded objects $*/G \to X$

Ideal solution to moduli problem:
$\Theta$-stratification where $X^{ss}$ and $Z^{ss}$ have GMS

Also includes variation of GIT quotient:

Thm: $\forall h \in NS(Y)^R_\Theta$, numerical invariant
$$\mu = h/\sqrt{\det b}$$
defines a $\Theta$-stratification
$$X = X^{ss} \cup U S,$$
where $X^{ss}(k)$ and $S_i^{ss}$ have GMS which are projective $\overline{Y}$

if $X$ is irreducible, as $l$ varies get birational modifications
**Meta-principal:**
Birational geometry of moduli spaces should be understood as variation of stability in some larger moduli problem with good moduli space

\[ q: X \rightarrow Y \]

Ex: Smyth classified all DM modular compactifications of \( \overline{M}_{g,n} \)

Would be nice to run MMP on \( \overline{M}_{g,n} \) by varying stability on moduli of all curves

Ex: Bayer-Macri prove that if

\[ X \rightarrowtail M_{g}^{hss}(S), \text{then} \]

\[ \text{CY k3} \quad X \in \{ \text{Bridgeland semistable} \} \]

\[ \{ \text{complexes on some twisted k3} \} \]

We will use this in the next lecture to study the local structure of flops

**Useful concept for constructing \( \Theta \)-stratifications**

**Def:** \( Y \) is \( \Theta \)-reductive if for any family over a discrete valuation ring, \( \text{Spec}(R) \rightarrow Y \), any filtration over the generic fiber extends uniquely to the special fiber.

**Thm A (HL):** Let \( X \) be a \( \Theta \)-reductive algebraic stack. Then a numerical invariant \( \mu \) defines a \( \Theta \)-stratification if and only if

1) Every unstable point has a unique HN filtration

2) In a bounded family \( \text{Spec}(A) \rightarrow Y \), only finitely many types of HN filtrations arise

**Rem:** main theorem of GIT is a special case
The notion of $\Theta$-reductive stack is useful for constructing good moduli spaces as well.

**Thm B (Alper-HL-Heinloth):** Let $\mathcal{X}$ be locally finite type with affine diagonal. Then $\mathcal{X}$ has a good moduli space if and only if

1) $\mathcal{X}$ is $\Theta$-reductive,

2) closed points of $\mathcal{X}$ have reductive automorphism groups, and

3) $\mathcal{X}$ has "unpunctured inertia"

These two theorems provide a program for analyzing general moduli problems, analogous to Keel-Mori theorem.

**Next time:** Discuss what Thm B means, and applications to Bridgeland semistable complexes.