Let $X_r$ be a del Pezzo surface of degree $9-r$, with $r \geq 3$, and let $L = k[x_{L_1}, \ldots, x_{L_N}]$ be the polinomial ring whose variables are indexed by the $(-1)$-curves $L_i$ on $X_r$. There is a natural grading on $L$ by the elements of the group $\text{Pic}(X_r) \cong \mathbb{Z}^{r+1}$, obtained by letting $\deg(x_{L_i}) = L_i$, and extending by multiplicativity.

There is a surjective ring homomorphism $\pi : L \to \text{Cox}(X_r)$. The morphism $\pi$ is graded with respect to the natural $\text{Pic}(X_r)$-grading on both $L$ and $\text{Cox}(X_r)$. Denote by $P$ the kernel of this morphism. Since $\pi$ is a graded ring map, the ideal $P$ is generated by homogeneous elements.

The purpose of this note is to prove that $P$ is generated by homogeneous elements whose $\text{Pic}(X_r)$-degree is a nef divisor.

**Lemma 1.** Let $D \in \text{Pic}(X_r)$ be an effective divisor. There are $0 \leq k \leq r$ disjoint $(-1)$-curves $E_1, \ldots, E_k$ on $X_r$, positive integers $n_1, \ldots, n_k$ and a nef divisor $N \in \text{Pic}(X_r)$ such that

1. $D = n_1E_1 + \ldots + n_kE_k + N$;
2. for all $i \in \{1, \ldots, k\}$ we have $E_i \cdot N = 0$;

Moreover, this decomposition is unique: the $(-1)$-curves $E_i$ and the positive integers $n_i$ are characterized by the property that $E_i \cdot D = -n_i$.

**Remark.** In case $k \geq 1$, a divisor satisfying the conditions in the conclusion of the lemma is clearly not nef: it suffices to intersect it with $E_1$ to get a negative intersection number with an effective curve.

**Proof.** Proceed by induction on the anticanonical degree $d := -K_{X_r} \cdot D$ of $D$. If $d = 1$, then an effective divisor $D$ on $X_r$ is either a $(-1)$-curve, or $r = 8$ and $D = -K_{X_8}$. In both cases the statement of the lemma is clear.

Suppose that $d \geq 2$. If for every $(-1)$-curve $E \subset X_r$ we have $E \cdot D \geq 0$, then $D$ is a nef divisor, and we may let $k = 0$. Together with the remark following the statement of the lemma, this concludes this case. Thus we may reduce to the case in which $D$ is effective and there is a $(-1)$-curve $E \subset X_r$ such that $E \cdot D < 0$. Since the divisor $D$ is effective, this means that $E$ is an irreducible component of $D$ and therefore that $D - E$ is an effective divisor. Since the anticanonical degree of $D - E$ is $d - 1$, by the inductive hypothesis we know that $D - E = m_1E_1 + \ldots + m_hE_h + N$, where $h \geq 0$, $m_i > 0$, the $E_i$'s are divisor classes of disjoint $(-1)$-curves on $X_r$ and $N$ is a nef divisor on $X_r$, such that $N \cdot E_i = 0$. Thus we have

$$D = E + n_1E_1 + \ldots + n_hE_h + N$$

Since $D \cdot E < 0$ and $N \cdot E \geq 0$, it follows that either $E = E_i$, for some $i \in \{1, \ldots, h\}$, or $E_i \cdot E = N \cdot E = 0$. In either case, the first part of the lemma follows.

The uniqueness of the decomposition of $D$ is an immediate consequence of the obvious characterization of the $(-1)$-curves $E_i$'s as the only $(-1)$-curves with negative intersection with $D$. \hfill \Box

We may obtain the decomposition of lemma 1. as an application of the Zariski decomposition of the divisor $D$: consider the linear system $|D|$ on $X_r$. Let $E_1, \ldots, E_k$ be the irreducible components of dimension one in the base locus $B$ of $|D|$, and let $n_i$ be the multiplicity of $E_i$ in $B$. The divisor $N = D - (n_1E_1 + \ldots + n_kE_k)$ is nef, since it only has isolated points as
base locus. The only statements left to show are that $N \cdot E_i = 0$ and $E_i \cdot E_j = -\delta_{ij}$. These equalities follow from the fact that the base locus of the complete linear system associated to a nef divisor on a del Pezzo surface is either basepoint-free or has a unique basepoint, in the case $r = 8$ and $D = -K_{X_8}$.

Let $D$ be a divisor in $Pic(X_r)$. We denote by $L_D$ the vector space of homogeneous elements of the ring $L$ of $Pic(X_r)$–degree $D$. Observe that we may find a vector space basis of $L_D$ consisting of monomials of $L$. Using lemma 1. and its notation, we may write $D = n_1E_1+\ldots+n_kE_k+N$.

**Lemma 2.** With notation as above, we have $L_D = x_{E_1}^{n_1} \cdots x_{E_k}^{n_k} \cdot L_N$.

**Proof.** Proceed by induction on the anticanonical degree $d$ of $D$. If $d = 0$, there is nothing to prove. Suppose that $d \geq 1$ and let $m = x_{L_1}^{\ell_1} \cdots x_{L_N}^{\ell_N}$ be a monomial in $L_D$. There is nothing to prove if $D$ is either nef or not effective. Suppose therefore that $D$ is effective but not nef. By the definition of the $Pic(X_r)$–grading on $L$, the degree $D$ of $m$ can be written as $\ell_1 L_1 + \ldots \ell_N L_N$. We may assume also that the numbering of the $(-1)$–curves $L_i$’s is such that $L_1 = E_1$. Since $D \cdot E_1 < 0$, the exponent $\ell_1$ of $x_{L_1}$ cannot be zero, and thus $m = x_{E_1} m'$, where $m'$ is a monomial in $L(D-E_1)$. By construction, $D-E_1 = (n_1-1)E_1 + n_2 E_2 + \ldots + n_k E_k + N$ is an effective divisor of anticanonical degree $d-1$. By the inductive hypothesis, $m' = x_{E_1}^{n_1-1} x_{E_2}^{n_2} \cdots x_{E_k}^{n_k} n$, where $n \in L_N$. Thus $m = x_{E_1}^{n_1} x_{E_2}^{n_2} \cdots x_{E_k}^{n_k} n$ and the lemma follows. \(\square\)

**Corollary 1.** The ideal $P$ is generated by the homogeneous elements of nef $Pic(X_r)$–degree.

**Proof.** This is clear, thanks to the previous lemma and the fact that the ideal $P$ is generated by homogeneous elements. \(\square\)