Cascades on interdependent networks

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A collection of interacting, dynamic networks form the core of modern society.

Networks:

- **Transportation Networks/Power grid** (distribution/collection networks)
- **Biological networks** - protein interaction - genetic regulation - drug design
- **Computer networks**
- **Social networks** - Immunology - Information - Commerce

- E-commerce → WWW → Internet → Power grid → River networks.
- Biological virus → Social contact network → Transportation nets → Communication nets → Power grid → River networks.
Critical infrastructure

From Peerenboom et. al.
Moving to systems of interdependent networks
What are the simplest, useful, *abstracted* models?

- What are the *emergent new properties*?
  - Host-pathogen interactions
  - Phase transition thresholds

- What features confer resilience in one network while *introducing vulnerabilities* in others?

- How do *demands* in one system shape the performance of the others? (e.g., demand informed by social patterns of communication)

- How do *constraints* on one system manifest in others? (e.g., River networks shape placement of power plants)

- **Coupling of scales across space and time** / co-evolution.
System of two networks

Connectivity for an individual node

- Degree distribution for nodes in network $a$: $p^a_{k_a k_b}$
- For the the system: $\{p^a_{k_a k_b}, p^b_{k_a k_b}\}$
- Generating functions to calculate properties of the ensemble of such networks.
Divide nodes initially into two groups (A and B):

- Add internal $a$-$a$ edges with rate $\lambda$.
- Add internal $b$-$b$ edges with rate $\lambda/r_1$, with $r_1 > 1$.
- Add intra-group $a$-$b$ edges with rate $\lambda/r_2$, with $r_2 > 1$, $r_2 \neq r_1$.

What happens? (Anything different?)
Wiring which respects group structures percolates earlier!

(Also tradeoffs between sparser and denser subnetworks.)

- Probability distribution for node degrees: \( \{ p^a_{k_a k_b}, p^b_{k_a k_b} \} \)
- Generating functions to calculate properties of the ensemble of such networks.
Consider two coupled random graphs.

Nodes fail (removed either in a targeted or random manner).

Following an **iterative removal process**, small failures can lead to massive cascades of failure of the networks themselves.

**Surprising**: What confers resilience to individual network (broad-scale degree distribution) may be a weakness for randomly coupled networks.
Approximated as power law $P_k \propto k^{-\gamma}$
$P_k \sim k^{-\gamma}$, the first two moments

(Note: $\gamma > 1$ required for $\sum_k P_k = 1$)

- First moment (Mean degree):
  \[
  \langle k \rangle = \sum_{k=1}^{\infty} k p_k \approx \int_{k=1}^{\infty} k p_k \, dk
  \]

  Diverges (i.e., $\langle k \rangle \to \infty$) if $\gamma \leq 2$.

- Second moment:
  \[
  \langle k^2 \rangle = \sum_{k=1}^{\infty} k^2 p_k \approx \int_{k=1}^{\infty} k^2 p_k \, dk
  \]

  Diverges (i.e., $\langle k^2 \rangle \to \infty$) if $\gamma \leq 3$.

- Many results follow for $2 < \gamma < 3$ since $\langle k \rangle / \langle k^2 \rangle \to 0$
Consequences of $p(k) \sim k^{-\gamma}$ for networks

- Most nodes are leaves (degree 1): *Network connectivity very robust to random node removal.*
- High degree nodes are hubs: *Network connectivity very fragile to targeted node removal.*

![Exponential vs. Scale-free networks](image)

- Epidemic spreading on the network (contact process):
  if $2 < \gamma < 3$, then $\langle k \rangle / \langle k^2 \rangle \to 0$ and $\lim_{n \to 0} \text{epidemic threshold} \to 0$.

(Buldyrev et al find broad scale more fragile for their particular cascade dynamics)
Dynamical processes on interdependent networks
Motivation: interconnected power grids


Power grid: a collection of interdependent grids. (Interconnections built originally for emergencies.)

Blackouts cascade from one grid to another (in a non-local manner).

Building more interconnections (Fig: planned wind transmission).

Increasingly distributed

Source: NPR
Motivation cont.: interconnected power grids

What is the effect of interdependence on cascades?

It is thought power grids organize to a "critical" state – power law distribution of blackout sizes – maximize profits while fearing large cascades.

Source: NPR
Sandpile models: “Self-organized criticality”

- Drop grains of sand ("load") randomly on nodes.
- Each node has a threshold for sand.
- Load > threshold $\Rightarrow$ node topples = sheds sand to neighbors.
- These neighbors may topple. And their neighbors. And so on.
- Cascades of load/stress on a system.

The classic Bak-Tang-Wiesenfeld sandpile model:

(Neuronal avalanches, banking cascades, earthquakes, landslides, forest fires, blackouts...)

- Finite square lattice in $\mathbb{Z}^2$
- Thresholds 4
- Open boundaries prevent inundation

Avalanche size follows power law distribution $P(s) \sim s^{-3/2}$
Sandpile model on arbitrary networks:

- Thresholds = degrees (shed one grain per neighbor)
- Boundaries: shedded sand are deleted independently with probability $f \approx 10/N$
- Mean-field behavior ($P(s) \propto s^{-3/2}$) robust. (Goh et al. PRL 03, Phys. A 2004/2005, PRE 2005. PLRGs with $2 < \gamma < 3$ not mean-field.)

Sandpiles on interacting networks:

- Sparse connections between random graphs.
- Configuration model with multi-type degree distribution.
Sparsely coupled networks

Two-type network: $a$ and $b$.

Degree distributions: $p_a(k_a, k_b), p_b(k_a, k_b)$

$$p_a(k_a, k_b) = \text{fraction of } a\text{-nodes with } k_a, k_b \text{ neighbors in } a, b.$$ 

Configuration model: create degree sequences until valid (even total intra-degree, same number of inter-edge stubs), then connect edge stubs at random.
Measures of avalanche size

- **Topplings:**
  Drop a grain of sand. How many nodes eventually topple?

  **Avalanche size distributions:** \( s_a(t_a, t_b), s_b(t_a, t_b) \)
  
  e.g., \( s_a(t_a, t_b) = \text{chance an avalanche begun in a topples } t_a \text{ many } a\text{-nodes, } t_b \text{ many } b\text{-nodes.} \)

  To study this, we need a more basic distribution...

- **Sheddings:**  Drop a grain of sand. How many grains are eventually shed from one network to another?

  **Shedding size distributions:** \( \rho_{od}(r_{aa}, r_{ab}, r_{ba}, r_{bb}) \)
  
  = \text{chance a grain shed from network o to d eventually causes } r_{aa}, r_{ab}, r_{ba}, r_{bb} \text{ many grains to be shed from } a \rightarrow a, a \rightarrow b, b \rightarrow a, b \rightarrow b

  *Approximate shedding and toppling as multi-type branching processes.
Cascades in networks $\approx$ branching processes if they’re *tree-like*.

Power grids are fairly *tree-like*:

<table>
<thead>
<tr>
<th></th>
<th>clustering coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power grid in SE USA</td>
<td>0.01</td>
</tr>
<tr>
<td>Similar Erdős-Rényi graph</td>
<td>0.001</td>
</tr>
</tbody>
</table>

Sandpile cascades on *interacting* networks $\approx$ a *multitype* branching process.
Overview of the calculations

From degree distribution to avalanche size distribution:

Input: degree distributions $p_a(k_a, k_b), p_b(k_a, k_b)$

$\Downarrow$ compute

shedding branching distributions $q_{aa}, q_{ab}, q_{ba}, q_{bb}$

$\Downarrow$ compute

toppling branching distributions $u_a, u_b$

$\Downarrow$ plug in

toppling branching generating functions $U_a, U_b$

$\Downarrow$ plug in

equations for avalanche size generating functions $S_a, S_b$

$\Downarrow$ solve numerically, asymptotically

Output: avalanche size distributions $s_a, s_b$
Example:

\[ q_{ab}(r_{ba}, r_{bb}) := \text{the branch (children) distribution for an } ab\text{-shedding}. \]

Probability a single grain shed from \(a\) to \(b\) results in \(r_{ba}\) \(a\)-sheddings and \(r_{bb}\) \(b\)-sheddings.
Shedding branch distributions $q_{od}$

The crux of the derivation

$q_{od}(r_{da}, r_{db}) :=$ chance a grain of sand shed from network $o$ to $d$ topples that node, sending $r_{da}, r_{db}$ many grains to networks $a, b$.

$$q_{od}(r_{da}, r_{db}) = \frac{r_{do} p_d(r_{da}, r_{db})}{\langle k_{do} \rangle} \frac{1}{r_{da} + r_{db}}$$

for $r_{da} + r_{db} > 0$.

- **I**: chance the grain lands on a node with degree $p_d(r_{da}, r_{db})$ (Edge following: $r_{do}$ edges leading from network $o$.)

- **II**: empirically, sand on nodes is $\sim \text{Uniform}\{0, ..., k - 1\}$

- Chance of no children $= q_{od}(0, 0) := 1 - \sum_{r_{da} + r_{db} > 0} q_{od}(r_{da}, r_{db})$ (Probability a neighbor of any degree sheds, properly weighted.)

- Chance at least one child $= 1 - q_{od}(0, 0)$. 
I. Edge following probability: single network

- Degree distribution, $P_k$, with G.F. $G_0(x) = \sum_k P_k x^k$.
- Probability of following a random edge to a node of degree $k$: $q_k = kP_k / \sum_k kP_k$, with G.F. $G_1(x) = \sum_k q_k x^k$.
- (“Contact immunization” strategy used by CDC.)
- Generating function “self consistency” construction.
  $H_1(x)$: G.F. for dist in comp size following random edge

\[
H_1(x) = xq_0 + xq_1 H_1(x) + xq_2 [H_1(x)]^2 + xq_3 [H_1(x)]^3 \cdots
\]

\[
= xG_1(H_1(x))
\]

(c.f. Newman, Strogatz, Watts *PRE* 2001.)
A node that just toppled is actually less likely to topple on the next time step. (prob zero sand $\neq 1/k$)
Toppling branch distributions \( u_a, u_b \)
shedding branch distributions \( q_{od} \sim \) toppling branch distributions \( u_a, u_b \)

**Key:** a node topples iff it sheds at least one grain of sand.

Probability an \( o \) to \( d \) shedding leads to at least one other shedding: \( 1 - q_{od}(0,0) \). Probability a single shedding from an \( a \)-node yields \( t_a, t_b \) topplings:

\[
u_a(t_a, t_b) = \sum_{k_a = t_a, k_b = t_b}^{\infty} p_a(k_a, k_b) \text{Binomial}[t_a; k_a, 1 - q_{aa}(0,0)] \cdot \text{Binomial}[t_b; k_b, 1 - q_{ab}(0,0)].
\]

(e.g., \( k_a \) neighbors, \( t_a \) of them topple, each topples with prob \( 1 - q_{aa}(0,0) \).)

Associated generating functions: \( U_a(\tau_a, \tau_b), U_b(\tau_a, \tau_b) \).
Summary of distributions and their generating functions

<table>
<thead>
<tr>
<th></th>
<th>distribution</th>
<th>generating function</th>
</tr>
</thead>
<tbody>
<tr>
<td>degree</td>
<td>$p_a(k_a, k_b), p_b(k_a, k_b)$</td>
<td>$G_a(\omega_a, \omega_b), G_b(\omega_a, \omega_b)$</td>
</tr>
<tr>
<td>shedding branch</td>
<td>$q_{od}(r_{da}, r_{db})$</td>
<td></td>
</tr>
<tr>
<td>toppling branch</td>
<td>$u_a(t_a, t_b), u_b(t_a, t_b)$</td>
<td>$U_a(\tau_a, \tau_b), U_b(\tau_a, \tau_b)$</td>
</tr>
<tr>
<td>toppling size</td>
<td>$s_a(t_a, t_b), s_b(t_a, t_b)$</td>
<td>$S_a(\tau_a, \tau_b), S_b(\tau_a, \tau_b)$</td>
</tr>
</tbody>
</table>

Self-consistency equations:

$$S_a = \tau_a U_a(S_a, S_b), \quad (1)$$
$$S_b = \tau_b U_b(S_a, S_b). \quad (2)$$

Want to solve (1), (2) for $S_a(\tau_a, \tau_b), S_b(\tau_a, \tau_b)$. Coefficients of $S_a, S_b =$ avalanche size distributions $s_a, s_b$.

In practice, Eqs. (1), (2) are transcendental and difficult to invert.
Numerically solving $\mathbf{\hat{S}}(\mathbf{\tau}) = \mathbf{\tau} \cdot \mathbf{U}(\mathbf{\hat{S}}(\mathbf{\tau}))$

Methods for computing $s_a, s_b$ for small avalanche size:

**Method 1:** Iterate starting from $S_a = S_b = 1$; expand.

**Method 2:** Iterate symbolically; use Cauchy’s integration formula

\[
S_a(t_a, t_b) = \frac{1}{(2\pi i)^2} \int \int_D \frac{S_a(\tau_a, \tau_b)}{\tau_a^{t_a+1} \tau_b^{t_b+1}} d\tau_a d\tau_b,
\]

where $D \subset \mathbb{C}^2$ encloses the origin and no poles of $S_a$.

**Method 3:** Multidimensional Lagrange inversion (IJ Good 1960):

\[
S_a = \sum_{m_a, m_b=0}^{\infty} \frac{\tau_a^{m_a} \tau_b^{m_b}}{m_a! m_b!} \left[ \frac{\partial^{m_a+m_b}}{\partial \kappa_a^{m_a} \partial \kappa_b^{m_b}} \left\{ h(\vec{\kappa}) U_a(\vec{\kappa})^{m_a} U_b(\vec{\kappa})^{m_b} \left\| \frac{\delta^\nu - \kappa_\mu}{U_\mu} \frac{\partial U_\mu}{\partial \kappa_\mu} \right\| \right\} \right]_{\vec{\kappa}=0},
\]

if the types $\mu, \nu \in \{a, b\}$ have a positive chance of no children.

- Unfortunately for large avalanches need to use simulation.
  (Asymptotic approximations used for isolated networks do not apply.)
Plugging in degree distributions: A real world example

Two geographically nearby power grids in the southeastern US.

<table>
<thead>
<tr>
<th></th>
<th>Grid c</th>
<th>Grid d</th>
</tr>
</thead>
<tbody>
<tr>
<td># nodes</td>
<td>439</td>
<td>504</td>
</tr>
<tr>
<td>$\langle k_{\text{int}} \rangle$</td>
<td>2.4</td>
<td>2.9</td>
</tr>
<tr>
<td>$\langle k_{\text{ext}} \rangle$</td>
<td>0.02</td>
<td>0.01</td>
</tr>
<tr>
<td>clustering</td>
<td>0.01</td>
<td>0.08</td>
</tr>
</tbody>
</table>

8 links between these two distinct grids. Different average internal degree $\langle k_{\text{int}} \rangle$. Long paths. (Low clustering – approximately locally tree-like.)
A canonical idealization: Random regular graphs

Two random $z_a$-, $z_b$-regular graphs with “Bernoulli coupling”: each node gets an external link independently with probability $p$. These $\approx$ power grids.

$$U_a(\tau_a, \tau_b) = \frac{(p - p\tau_a + (z_a + 1)(\tau_a + z_a - 1))^{z_a}(1 + p(\tau_b - 1) + z_b)}{(z_a + 1)^{z_a}z_a^{z_a}(z_b + 1)}$$
Matching theory and simulation (for small’ish avalanches)

Plot marginalized avalanche size distributions

\[ s_a(t_a) \equiv \sum_{t_b \geq 0} s_a(t_a, t_b), \quad s_a(t_b) \equiv \sum_{t_a \geq 0} s_a(t_a, t_b), \quad \text{etc.} \]

in simulations, branching process.

Regular(3)-Bernoulli(\(p\))-Regular(10)  
Power grids \(c, d\).
Main findings: For an individual network, optimal $p^*$

- (Blue curve) Initially increasing $p$ decreases the largest cascades started in that network (second network is reservoir for load).
- (Red curve) Increasing $p$ increases the largest cascades inflicted from the second network (two reasons: new channels and greater capacity).
- (Gold curve) Neglecting the origin of the cascade, the effects balance at a stable critical point, $p^* \approx 0.1$. (Reduced by 75% from $p=0.001$ to $p=0.1$)
Main findings: Individual network, “Yellowstone effect”

Supressing largest cascades amplifies small and intermediate ones! (Supressing smallest amplifies largest (Yellowstone and Power Grids*))

- To suppress smallest, isolation $p = 0$.
- To suppress intermediate (10% of system size) either $p = 0$ or $p = 1$.
- To suppress cascades > 25% of system size then $p = p^* \approx 0.11$.

*Dobson I, Carreras BA, Lynch VE, Newman DE Chaos, (2007).*
Main findings: System as a whole

More interconnections fuel larger system-wide cascades.

- Each new interconnection adds capacity and load to the system (Here capacity is a node’s degree, interconnections increase degree)

- Test this on coupled random-regular graphs by rewiring internal edges to be spanning edges (increase interconnectivity without increasing degree). No increase in the largest cascades.

- Inflicted cascades (Red curve) increase mostly due to increased capacity.

- So an individual operator adding edges to achieve $p^*$ may inadvertently cause larger global cascades.
Larger cascades from increased interconnections: A warning sign?

- Financial markets
- Energy transmission systems

Unless the coupled grids are identical, only one will be able to achieve its $p^*$.

- Coupled $z_a \neq z_b$ regular random graphs (branching process and simulation).

$$\frac{\langle s_a \rangle_b}{\langle s_b \rangle_a} = \frac{1 + z_a}{1 + z_b}$$

If $z_b > z_a$ inflicted cascades from $b$ to $a$ larger than those from $a$ to $b$.

(An arm’s race for capacity?)
Some interconnectivity can be beneficial, but too much is detrimental. Stable optimal levels are possible.

From perspective of isolated network, seek optimal interconnectivity $p^*$. This equilibrium will be frustrated if the two networks differ in their load or propensity to cascade.

Tuning $p$ to suppress large cascades amplifies to occurrence of small ones. (Likewise, suppressing small, amplifies large.)

Additional capacity and overall load from new interconnections fuels larger cascades in the system as a whole.

What might be good for an individual operator (adding edges to achieve $p^*$), may be bad for society.
Possible extensions – Real power grids

- Expand multi-type processes to encode for different types of nodes (buses, transformers, generators)

- Linearized power flow equations – cascades in real power grids are non-local: e.g. fig: 3 to 4, 7 to 8

- Game theoretic/economic consideration (we assume adding connections is cost-free)

(1996 Western blackout NERC report)

(Power grids as "critical" – Balancing profit and fear of outages)
Possible extensions

Teams and social networks
- Tasks (sand) arriving on people (nodes)
- Each person has a capacity for tasks: sheds once overloaded
- Coupling to a second social network (team) can reduce large cascades

Amplifying cascades
- Encourage adoption of new products
- Snowball sampling

Airline networks
- Different carriers accepting load (bumped passengers)
Other types of cascades, not just than sandpiles

- Watt’s threshold model: “topple” is some fraction $\phi$ of your neighbors have “toppled” (rather than “toppling”, Watt’s think of cascades in adopting a new product).
  – Harder to “topple” nodes of high degree.

- Kleinberg: rather than thresholds, diminishing returns (concave / sub-modular utility)

Note Author Summary for high-level overview.