Feb. 27

Read: Graver, Synchronization (sp?)
Chapter 1 to 2.4

local/infinitesimal theory
mostly bar frameworks
first order theory

Trivialities

Let \( p = (p_1, \ldots, p_n) \) be a \( p \) in \( \mathbb{E}^d \) configuration

**Def.** A congruence of \( \mathbb{E}^d \) is a transformation \( T: \mathbb{E}^d \to \mathbb{E}^d \) where \( T(p_i) = A p_i + b \)
where \( A = \text{orthogonal matrix} \) and \( b = \text{vector in } \mathbb{E}^d \).

Orthogonal means \( \mathbf{A}^T \mathbf{A} = \mathbf{A} \mathbf{A}^T = \mathbf{I} \).

\[
|T_{p_i} - T_{p_j}|^2 = |A p_i + b - A p_j - b|^2 = |A p_i - A p_j|^2 = |A (p_i - p_j)|^2 = (p_i - p_j)^T A^T A (p_i - p_j) = (p_i - p_j)^T (p_i - p_j) = |p_i - p_j|^2
\]

So distances are preserved.

**Fact:** If \( T: \mathbb{E}^d \to \mathbb{E}^d \) preserves distances, then it is a congruence.

**Proof:** [No exercise.]

Let $G(p)$ be a (tensegrity) framework. (Can think of this as a Sundar framework.)

**Def:** A path $p(t) = (p_1(t), \ldots, p_n(t))$, $0 \leq t \leq 1$, where $p(0) = p$ is called a flex of $G(p)$ if

$$p(t) 
= p
\leq G(p).$$

**Def:** If $p(t)$ is a flex, we say it is trivial if there is a family $T_k : IE^d \to IE^d$ where $T_0 = \text{identity}$, $T_k$ is a congruence, and $T_k(p_i) = p_i(t)$ for $0 \leq t \leq 1$. (Normally, think of $T_k$ being continuous.)

**Def:** We say $G(p)$ is rigid if every continuous flex of $p(t)$ is trivial.

**Examples:**

- **rigid**

- **rigid in $\mathbb{R}^2$**
  (not rigid in $\mathbb{R}^3$)

- not rigid in $\mathbb{R}^2$

- not rigid in $\mathbb{R}^2$

- rigid in $\mathbb{R}^2$
  (although engineer would not call it rigid)

- **drwll...**
Theorem: Let \( G(p) \) be a tensegrity framework in \( \mathbb{R}^d \). Then the following are equivalent.
1. Every continuous flex \( p(t) \) is trivial.
2. Every analytic flex \( q(t) \) of \( p(t) \) is trivial.
3. There is a \( \delta > 0 \) such that if \( |q - p| < \delta \) and \( G(q) \leq G(p) \) then \( q \) is congruent to \( p \). (\( T_p q_i \), \( T = \text{congruence} \))

Proof (Hint): The members constraints
\[ |q_i - q_j|^2 \leq \sum_{i \neq j} |p_i - p_j|^2 \]

Variable \( \uparrow \)
Fixed \( \downarrow \)

Polynomial inequalities and equalities defined on the configuration space \( \mathbb{R}^d \) (the space of configurations).

Such a finite set defined by polynomial inequalities and equalities is called a semi-algebraic set.

An algebraic set would be one defined by only equalities.

In Milnor's Singular Points on Complex Hypersurfaces, it is shown that in the neighborhood of any point \( q \) in \( X \), a semi-algebraic set, there is an analytic fibration. (?) This provides the analytic parametrization.

End of hint for proof.

If \( p(t) \) is an analytic flex of \( G(p) \) then
we can take the derivative.

\[ |p_i(t) - p_j(t)|^2 = \sum_{j=1}^{n} |p_i - p_j|^2 \]

\[ \epsilon_{i,j,k} = \begin{cases} \text{bar} & \text{if \( i,j,k \) are \( \text{non-collinear} \)} \\ \text{strut} & \text{otherwise} \end{cases} \]

Differentiate, at \( t=0 \):

\[ 2 \left( p_i(t) - p_j(t) \right) \cdot \left( p_i'(t) - p_j'(t) \right) = 0 \]

where \( p_i(t) = \frac{d}{dt} p_i(t) \) at \( t=0 \).

\[ (p_i - p_j) \cdot (p_i' - p_j') \leq 0 \]

Any \( p' \) where

\[ + p' = (p_1', \ldots, p_n') \]

is called an infinitesimal flex of \( G(p) \). So this is a little vector field associated with \( G(p) \).

**transformation**

\[ \begin{array}{c}
\text{translation} \\
\uparrow
\end{array} \]  

\[ \begin{array}{c}
\text{rotation} \\
\uparrow
\end{array} \]

or translate a different direction

\[ \leftarrow \]
Both are rigid motions. Only trivial motions.

\[ \cdot \rightarrow 2 \text{ dimensional flexes} \]
\[ \triangle (2 \text{ dimensions of possible flexes}) \]
But this object had 3 dimensions of flexes \( \triangle \) for translations + 1 for rotations.

The reason these don't have the same number of dimensions is that the point \( \cdot \) has too small a dimension \((-\text{affine span not } \mathbb{R}^2)\).

More Trivialities:

Def.: Then a trivial infinitesimal flex of \( GL(p) \) is the time 0 derivative of an analytic congruence \( T_\varepsilon \) where \( T_0 = I \).

Remember \( T_\varepsilon (p_i) = A_\varepsilon p_i + b_\varepsilon \) where \( A_\varepsilon A_\varepsilon^T = I \).

Trivial flex \( \frac{dA_\varepsilon}{dt} p_0 + db_\varepsilon = 0 \)

If \( A_\varepsilon \) is an analytic family of orthogonal matrices, \( A_\varepsilon A_\varepsilon^T = I \).

\[ 0 = \frac{d}{dt} (A_\varepsilon^T A_\varepsilon) = (\frac{d}{dt} A_\varepsilon^T) A_\varepsilon + A_\varepsilon (\frac{d}{dt} A_\varepsilon) \]
Let \( \frac{d}{dt} A(t) \big|_{t=0} = S \)

Then
\[ A(t) \big|_{t=0} = 0 = S^T + S \]
\[ S^T = -S \]

So \( S \) is a skew-symmetric matrix.

So for \( p_i' \) to be a trivial infinitesimal flex, we have that

\[ p_i' \sim S p_i + b \]

A constant corresponds to infinitesimal translation.

\[ \frac{d^2 - d}{2} = \frac{d(d+1)}{2} \]

Dimensional space corresponds to infinitesimal translation.

Total dimension of trivial infinitesimal flexes is

\[ \frac{d^2 - d}{2} = \frac{d(d+1)}{2} = \frac{d+1}{2} \]

(assuming our object does not have two low of dimension such as the point \( \in \R^2 \)).

For \( d=3 \) we can write

\[ S p_i = S \times p_i \]

\[ S = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix} \]

Can use same formula in planes, if you like.
Def: A (tensegrity) framework $G(p)$ is infinitesimally rigid if every infinitesimal flex $p'$ of $G(p)$ is trivial.

Examples:

inf. rigid in $\mathbb{E}^2$

inf. rigid in $\mathbb{E}^2$

inf. rigid in $\mathbb{E}^2$, not globally rigid.

inf. rigid (we'll prove later)

not inf. rigid, despite being rigid.

not inf. rigid, not globally rigid just take some inner rigid.
Connect all +'s to all -'s

$K_{3,3}(p)$

But if it doesn't lie on a conic, then it would be inf. rigid.

Let $\mathcal{E}^n$ be the space of all configurations of $n$ points in $\mathbb{R}^d$.

$p \in \mathcal{E}^n \Rightarrow \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$

Let $\mathcal{E}^e$ be the "space" of metrics for a graph $G$ where $e = \#$ of members of $G$.

Define $f: \mathcal{E}^n \rightarrow \mathcal{E}^e$ by

$p = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \rightarrow \begin{bmatrix} (p_1-p_2)^2 \\ \vdots \\ (p_{e-1}-p_e)^2 \end{bmatrix} \leq \varepsilon$, $i,j \leq 3$ member of $G$

So $G(p)$ is bar framework is rigid if

$f^{-1}(\emptyset) \cap \text{Up} = \emptyset$, $q \sim p_3 \cap \text{Up}$

where $\text{Up} = \text{nbhd of } p$ in $\mathcal{E}^n$.
We compute $\nabla f$, the differential of $f$ at $p$.

Let $\phi_i \in \mathcal{E}^n, (\text{Think \ infinitely \ small \ (flexible)})$ in $\mathcal{E}^n$.

Compute: $\frac{d}{dt} \phi_i(p(t)) \bigg|_{t=0} = df_{p_i}(v_i p_i)$

$p(t) = p + \epsilon p_i$

Hence:

$$
(p_i(t) - p_j(t))^2 = (p_i - p_j + \epsilon(p_i - p_j))^2
$$

$$
\frac{d}{dt} = [O(p_i - p_j) + t(O(p_i - p_j) \cdot (p_i - p_j))]
$$

$$
= (p_i - p_j)^2 + 2(\epsilon p_i \cdot (p_i - p_j))
+ \epsilon^2 (p_i - p_j)^2
$$

Consideration: $O$ is a motion unit

$$
d f_{p_i}(p_i) = 2 \begin{bmatrix}
1 & & \\
& (p_i - p_j) & \\
& & (p_i - p_j')
\end{bmatrix}
$$

In terms of standard coordinates $\mathcal{E}^n$ the matrix $[\nabla f(p)]$ is

$$
\begin{bmatrix}
0 & \cdots & 0 & (p_i - p_j)^T 0 & \cdots & 0 & (p_i - p_j')^T 0 & \cdots & 0
\end{bmatrix}
$$

and

$$
2 \nabla f(p) p_i = (p_i - p_j)'(p_i - p_j')
$$
Define $R(p)$ be rigidity matrix.

Note for a bar framework, $G(p)$ is infinitesimally rigid iff the kernel of $R(p)$ consists only of trivial infinitesimal flexes.

What is the dimension of trivial infinitesimal flexes of a configuration $\langle p_1 \ldots p_n \rangle$? 

$$\dim_\mathbb{R} \ker (S \text{ skew symmetric, } b = \text{transl.}) = \frac{d(d+1)}{2}$$

But problem if dimension is not same as space you are in. Ex: a point.

Skew matrices $\times$ translations $\rightarrow$ trivial flexes on a given configuration.

Claim: This map is onto one (had kernel 0) when $\dim \langle p \rangle = d$ (or $d-1$).

What happens to inf. rigidity when $\langle p \rangle \neq \mathbb{E}^d$?

Ex: $d = 3$

Have something in plane.

Only examples inf. rigid in $\mathbb{E}^3$. 
Theorem: If \( G(p) \) is a bar framework, \( \langle p \rangle \approx ED \), then \( G(p) \) is infinitesimally rigid if and only if the dimension of the kernel of \( R(p) \) is equal to \( \frac{d(d+1)}{2} \).

Corollary: Rank \( R(p) = nd - d(d+1) \)

So if \( G(p) \) is inf. rigid \( \frac{d(d+1)}{2} \)

(not a simplex) then \( e \approx nd - \frac{d(d+1)}{2} \)

\( d = 2 \quad e \approx 2n - 3 \)

\( d = 3 \quad e \approx 3n - 6 \)

('inf. rigidity should feel rigid')

\( n \) vertices all \( N \)'s

\( 3n - 6 \) edges

\( e \approx 3n - 4 \)

If you remove any edge, there is a nontrivial flex,
Hint: For #2:
\[ f : \mathbb{R}^d \rightarrow \mathbb{R}^d \]
\[ f(0) = 0 \]

\[ \text{let } \{ p_0, \ldots, p_k \} = p \text{ be the vertices of the simplex in } \mathbb{R}^d \]

Let \( p' = (p_0', \ldots, p_k') \) be an infinitesimal flex of \( G(p) \) where \( G \) is the complete graph.

\( p_0, \ldots, p_k \) is affine independent, \( \forall i \neq j \), \( p_i - p_0, \ldots, p_k - p_0 \) is linearly independent in \( \mathbb{R}^d \).

Trivial infinitesimal flexes of \( G(p) \) are a linear space, so if we replace \( p = (p_0, \ldots, p_k) \) with \( (p_0' = p_0', p_1' = p_1', \ldots, p_k' = p_k') \), we have not changed whether \( p' \) is trivial. \(< (p_0, \ldots, p_k) \rightarrow (p_0', \ldots, p_k') >\) is an infinitesimal translation.

Let us also assume \( p_0 = 0 \).

Then:
\[ (p_i - p_0) \cdot (p_i' - p_0') = p_i' \cdot p_i = 0 \]
\[ \forall i = 1, \ldots, k \]

and:
\[ (p_i - p_j) \cdot (p_i' - p_j') = 0 \]
\[ -p_i' \cdot p_j = -p_j \cdot p_i' \]
\[ p_i' \cdot p_j' = p_i \cdot p_j' \]
\[ \forall i, j = 1, \ldots, k \]
Let \( S : \mathbb{E}^d \rightarrow \mathbb{E}^d \) be given by \( S \mathbf{p}_i = \mathbf{p}_i' \).

\[ S \mathbf{p}_i \cdot \mathbf{p}_j = \mathbf{p}_i' \cdot \mathbf{p}_j = -\mathbf{p}_j' \cdot \mathbf{p}_i = -\mathbf{p}_i \cdot S \mathbf{p}_j \]

\( S \) is a linear transformation.

Claim: \( S \) can be extended to all of \( \mathbb{E}^d \) so that \( S \mathbf{v} \cdot \mathbf{w} = -\mathbf{v} \cdot S \mathbf{w} \), for all \( \mathbf{v}, \mathbf{w} \) in \( \mathbb{E}^d \).

Then \( S \) is skew symmetric.

(Exercise.)

Let \( A : \mathbb{E}^d \rightarrow \mathbb{E}^d \) be linear. \( G(p) \) is infinitesimally rigid in \( \mathbb{E}^d \).

\[ G(A \mathbf{p}) = G(A \mathbf{p}_1, A \mathbf{p}_2, \ldots, A \mathbf{p}_n) \]

Let \( \mathbf{p}' \) be an inf. flex of \( G(p) \).

\[ \mathbf{0} = (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}_i' - \mathbf{p}_j') \]

\[ \mathbf{0} = (A \mathbf{p}_i - A \mathbf{p}_j) \cdot (A \mathbf{p}_i' - A \mathbf{p}_j') \]

\[ = (\mathbf{p}_i - \mathbf{p}_j) \cdot (A \mathbf{p}_i' - A \mathbf{p}_j') \]

Assume \( A \) is invertible.

Test for triviality of an infinitesimal flex.

If \( G(p) \) is not a \( k \)-dimensional simplex, \( k < d \),
then \( \mathbf{p}' \) is a non-trivial flex of \( p \) iff

\[ (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}_i' - \mathbf{p}_j') \neq 0 \text{ for some } i \neq j. \]

If \( (\mathbf{p}_i - \mathbf{p}_j)' = (\mathbf{p}_i' - \mathbf{p}_j) = 0 \) \( \forall i, j \) then \( \mathbf{p}' \) is trivial.

i.e. \( K_n(p) \) is infinitesimally rigid in \( \mathbb{E}^d \).

This is true if affine span of \( p \) is \( d' \)-dimensional.

This can be shown by starting with a simplex + building.
March 8

n = # vertices = 8
e = # edges = 12

3n - 6 = 18 > 17
So not inf. rigid.

Note: with one extra bar inf. rigid. (?)\n
Cauchy (1813) Convex polyhedra are "unique" in the class of convex polyhedra.

m. Dehn student of Hilbert\n
Solved Hilbert's 3rd problem before it was stated (1916) Theory: Let \( P \) be a convex polyhedron with all its facets triangles and let \( G(p) \) be the bar framework formed using the vertices and edges of \( P \). Then \( G(p) \) is infinitesimally rigid in \( \mathbb{R}^3 \).

Examples:

First nontrivial example:

Proof: (of Dehn's theorem)

Note since \( P \) has all triangulated facets, \( 3n - 6 = e \) (from Euler formula).
Matrix multiplication - air plane flying into building

So the rows of the rigidity matrix \( R(p) \) are independent if and only if the rank of \( R(p) \) is 3n - 6 if \( G(p) \) is inf. rigid.

So we want to show rank \( R(p) = 3n - 6 \).

So we wish to show that the only solutions to \( WR(p) = 0 \) are \( w = 0 \) where \( w = (\ldots, w_ij, \ldots) \) on edge between \( i \to j \).

\[
WR(p) = (\ldots, w_ij, \ldots) \begin{bmatrix}
(P_i - P_j)^T & 0 & \ldots & (P_j - P_i)^T
\end{bmatrix}
\]

3 columns

\[
\sum_j w_ij (P_i - P_j)^T
\]

3 entries (columns)

\( w = 0 \) if \( R(p) \) is an equilibrium stress for \( G(p) \).

We need to show that \( w = 0 \) is the only equilibrium stress for our framework.

Pause to justify what is coming next.

Let's look at octahedron again (this time we look down on octahedron).

Only possibility for signs of stresses (or negative of this).
have to have this same sign pattern everywhere.

So octahedron must be inf. rigid.

Combinatorial

Lemma (Cauchy): Label each edge of a planar graph \( G \) (without loops or multiple edges) plus or minus \((+ or -)\). Then \( G \) has a vertex \( v \) that has no changes in sign or just 2 changes in sign around the cyclic order around a vertex (or stated: all the edges are seen cyclically.)

Proof:

Suppose that every vertex of \( G \) has at least 4 changes in sign. Look for a contradiction.

Let \( n \) = number of vertices of \( G \).
\( e \) = number of edges of \( G \).
\( f_i \) = number of \( i \)-face of \( G \) with \( i \)-edges.
\( f \) = total number of \( i \)-face of \( G \).

Euler: \( n - e + f = 2 \)

\( f = f_3 + f_4 + ... \)
\( 2e = 3f_3 + 4f_4 + ... \)

Put a \( + \) \( \times \) \( v \) between each \( + \) \( - \) at a vertex \( v \) of \( G \):

Let \( \phi \) be total number of \( v \)-checks (\( v \)).
Can only have 2 checks in triangular face

\[ 4n \leq e \leq 2f_3 + 4f_4 + 4f_5 + 6f_6 + 6f_7 + \ldots \]

Now we use algebra to get

\[ 4n = 8 - 4f + 4R \]

\[ = 8 - 4(\ell + \eta + \ldots) + 4(\frac{3}{2}f_3 + 2f_4 + \ldots) \]

\[ = 8 - 4(\ell + \eta + \ldots) + 2(3f_3 + 4f_4 + \ldots) \]

\[ = 8 + 2f_3 + 8f_4 \leq 2f_3 + 4f_4 + 4f_5 + 6f_6 + 6f_7 + \ldots \]

\[ \Rightarrow 8 + 6f_3 + 6f_4 + 6f_5 + 6f_6 + \ldots \leq 0 \]

All coefficients \( \geq 0 \).

\[ \text{(of lemma)} \]

Finishing the proof of Pach's Alexander Theorem:

If \( G(p) \) is not in \( \text{inf. rigid} \), there must be a \( \text{non-zero}\) after equilibrium stress \( \omega \) where \( \omega \cdot R(p) = 0 \). Apply combined toroidal lemma to the subgraph \( H \subseteq G \) with a \( \text{non-zero} \) stress. Then any reroute with at most 2 changes in sign is not in equilibrium. (This argument does not apply to triangulated polytopes with facets that are not \( \Delta \)'s.).
Projects?

- Bracing a square grid
- Bellows properties
- Cauchy mechanics: higher order rigidity
- Tension percolation
- Polyhedral combinatorics (relative min. rigidity)

Plan
Near Future: - Cauchy's original theorem (today)
- Generic rigidity
- Projective invariants
- Basic rigidity \( \leq \) int. rigidity
- Generic global rigidity

Theorem (Cauchy, 1813) Modern Version
Let \( h: \partial P \rightarrow \partial Q \) be a homeomorphism where \( P, Q \) are compact convex polyhedra in \( \mathbb{R}^3 \) where \( \partial P, \partial Q \) are their boundaries, and \( h \) restricted to each face of \( P \) is a congruence, then \( h \) extends to a congruence \( h: E^3 \rightarrow E^3 \), i.e., \( h(\partial P) = \partial Q \).

Proof:

Remark (Grünaume): This result should be stated carefully.

False statement: Let there be a 1-1 incidence preserving correspondence between the faces of \( P \) and \( Q \), two convex polytopes in \( \mathbb{R}^3 \) such that corresponding faces are congruent. Then \( P \) and \( Q \) are congruent.

False because the correspondence may not match up.

Homework: Give false example.
Proof: (of Cauchy's theorem)

For each edge $e$ of $P$, let $\Theta_e$ be the dihedral angle of $P$ at $e$. (Dihedral angle is the internal angle between two planes along the edge $e$.) Let $\Theta_{he}$ be the corresponding dihedral angle of edge $e$ in $\mathbb{Q}$.

To each edge associate a $+ \quad \Theta$ if $\Theta_e < \Theta_{he}$ and $- \quad \Theta$ if $\Theta_e > \Theta_{he}$.

We claim that for each vertex $P_i$ of $P$, there are either no signed $\Theta$'s or there are at least 4 sign changes. But $\Rightarrow$ impossible.

Claim: these not possible.

Let's look at a vertex $P_i$:

Suppose there are exactly 2 sign changes around $P_i$. Let $u, v$ be two points $u, v \not= P_i$ in $\mathbb{Q}$ adjacent to $P_i$ such that $\langle P_i, u \rangle, \langle P_i, v \rangle$ separate the signed $\Theta$'s from the $-\Theta$'s.
Choose a plane \( T \) parallel to a support plane of \( P \) through \( u \) and \( v \) such that \( P \) is on one side of \( T \) and other vertices of \( P \) are on the other side.

\[ C = T \cap \partial P \] is a simple closed curve.

Vertices of \( C \) correspond to the edges of \( P \) adjacent to \( P \).

One path on \( C \) from \( u \) to \( v \) consists of only \( + \) vertices and under \( h \) the angles of \( C \), must only increase or stay the same (and some increase).

So the Cauchy arm lemma implies \[ |h(u) - h(v)| > |u - v| \] (Note: Cauchy arm lemma works in any dimension)

Now apply a similar argument in going from \( Q \) to \( P \). We get \[ |u - v| > |h(u) - h(v)| \]. This is a contradiction.

This is why we need both \( P \) and \( Q \) to be convex.

If all signs at \( P \) are + or all - use a similar argument, create a contradiction.
So we can't have 2 changes in sign or no changes in sign. But the combinatorial lemma proved last time implies that all the edges have no sign attached, i.e., all dihedral angles $\Theta_e = 0$ (hence) this implies that it extends to a congruence of $\mathbb{R}^3$.

A.D. Alexandrov generalized Duh's rigidity result. Triangulate the surface $\partial P$, but there are no vertices in the interior of a 2-dimensional face of $P$.

Then, this bar framework is infinitesimally rigid, so therefore rigid.

Classic example: cube

\[ \text{\textbf{Weak Alexandrov's theorem}} \]

\[ \text{no new vertices} \]

\[ \text{Note: we can't just use proof of Duh's theorem} \]

\[ \text{Stronger version: subdivide in interior of edges} \]

\[ \text{etc.} \]

\[ \text{can use weak version to show the stronger version} \]

\[ \text{(if inf rigid, then new vertices must inf rigid, w/new vertices).} \]

\[ \text{Can generalize to more dimensions,} \]

\[ \text{Just need to triangulate 2-skeleton.} \]
Theorem: (Connelly)

Let $\mathcal{G}(p)$ be a bar framework coming from any triangulation of the boundary of $P$, a convex polytope, then $\mathcal{G}(p)$ is second-order rigid and therefore rigid.

Pre-stress stable

Next time: generic rigidity.
Coming events:
- Finish Laman theorem today
- Projective invariance of inf-rigidity
- Static Rigidity

Possible project: pebble game.

Laman's Theorem: \( G \) is generically rigid in \( \mathbb{R}^2 \) if and only if \( e \leq 2n - 3 \) for any subset of edges \( e' \) and their vertices \( n' \).

So we need to show:

Hinrichs operation

If

is generically rigid

is inf-rigid in this special position and therefore generically rigid

Noticed last time: average degree of \( G \) is \(< 4\). So \( G \) has a vertex of degree \( 1, 2, \) or \( 3 \).
Suppose graph $G'$ has vertex with degree 1:

$G'$ has $n' = n - 1$ vertices and $e' = e - 1$ edges.

So $2n' - 3 = 2(n - 1) - 3 = 2n - 5 = (2n - 3) - 2 = e - 2 = (e - 1) - 1 = e' - 1 < e'$.

So $2n' - 3 < e'$.

Thus we cannot have a vertex of degree 1.

Suppose graph $G''$ has vertex with degree 2:

$G''$ has $n'' = n - 1$ vertices and $e'' = e - 2$ edges.

So $2n'' - 3 = e''$.

So by induction on $n$, $G'$ is 2-generically rigid. By problem 3 in HW 6, $G$ is inf. rigid.

(Base case $n = 1$: generically rigid.)
degree 3 (hard case)
Use induction $n = k$ vertices.

We claim there is a subgraph for some $i, j \in 1, 2, 3$ $G'_{U1ij}$ is generically 2-rigid.

If this happens for any $i, j$ then the Hinneberg operation creates an inf. rigid $G$ and therefore $G$ is generically 2-rigid.

So assume all 3 cases yield a $G'_{U1ij}$ not generically 2-rigid.

So by induction we know that the "count" is wrong in each case. Let $G_{ij}$ be the subgraph of $G'$ with $e' > 2n^1 - 3$.

Each $G_{ij}$ must be generically 2-rigid because when you add the edge $e'_{ij}$ $e' > 2n^1 - 3$.

So in $G_{ij}$, edges = 2 vertices - 3.
So inductively generically 2-rigid.
Consider $G_{ij} \cup G_{jk}$

Claim: $G_{ij} \cup G_{jk} = \mathcal{E}_j^3$

If there is another vertex then $G_{ij} \cup G_{jk}$ is inf. rigid. Then adding $p_0$ and its three edges create a subgraph with $e = 2n - 3 + 1$.

So $G_{ij} \cup G_{jk} = \mathcal{E}_j^3$.

So we have

Now $G_{12} \cup G_{32} \cup G_{31}$ is inf. rigid.

So $e' = 2n' - 3$ for this union.

Adding $p_0$ and its 3 edges create a count $e = 2n - 3 + 1$ contradicting the hypothesis.

Thus some Henneberg can be done.

And we are done.

Projective invariance of infinitesimal rigidity.

Choose $p_0 \in \mathcal{E}_d$.

Connect $p_0$ by a bar to each $p_i \in \mathcal{E}_d$.

Let $G(p)$ be a framework. $\mathcal{E}_d < \mathcal{E}_d'$.
New framework. \( \hat{G}(\hat{p}) \)

\( \hat{p} = (p_0, p_1, \ldots) \)

Let \( p' \) be an inf. flex of \( G(p) \) in \( IE^d \).
Let \( p_0' = 0 \); we want to define \( \hat{p}' \) as a corresponding inf. flex of \( \hat{G}(\hat{p}) \).

\[
\begin{align*}
\hat{p} & \quad \hat{p}' \\
p & \quad p' \\
\hat{p}_i & \quad p_i'
\end{align*}
\]

Let \( e_d \) be \( 1 \) to \( IE^d \). We want to define \( \lambda_i \) so that \( (p_i' - p_0')(p_i' + \lambda_i e_{d+1} - p_0') = 0 \)

Define \( \lambda_i = \frac{-p_i' - (p_i - p_0)}{(p_i - p_0) \cdot e_{d+1}} \neq 0 \)

So bar constraints are satisfied and 
\( (p_i - p_j)(\hat{p}_i - \hat{p}_j) = (p_i - p_j)(p_i' - p_j') \)
so member constraints elsewhere are satisfied.

7. Prop: \( G(p) \) is inf. rigid in \( IE^d \) if and only if
\[
\hat{G}(\hat{p}) \text{ is inf. rigid in } IE^{d+1}.
\]

Would it be possible to have something inf. rigid in \( IE^d \) but not in \( IE^{d+1} \)?
Let $G(p)$ be a cone framework. Then replace

Let $p_0 = 0$. Replace $p_i$ with $k_i \cdot p_i$ where $k_i \neq 0$.

Then $G(p) + G(p')$ are both inf. rigid or both inf. flexible.

We can assume $p_0' = 0$. \\
Bar constraints $(p_i - p_0) \cdot (p_i' - p_0') = p_i \cdot p_i' = \frac{1}{\lambda^2} (\lambda p_i) \cdot (\lambda p_i')$

$$= \frac{1}{\lambda^2} (\lambda p_i \cdot p_i') = 0.$$ \\
Other constraints $(p_i - p_j) \cdot (p_i' - p_j') = -p_i \cdot p_j' - p_j \cdot p_i'$

If each strut remains col. or str. when $\lambda_i \lambda_j > 0$ \\
Reverse roles when $\lambda_i \lambda_j < 0$

Projective invariance follows modulo this remaining members.
Question: Does there exist a generically rigid graph, embedded in the plane (no crossings) with \( \Delta \), that is generically rigid in the plane.

No, needs to have a \( \Delta \), to be generically rigid need \( e \geq 2v - 3 \)

\[ 2e = 4f \]
\[ e = 2f = 2(2 - v + e) = 4 - 2v + 2e \]
\[ 2v = 4 + e \]
\[ 2v - 3 = 1 + \# e > e \]

We continue the discussion of first order rigidity with

**Static Rigidity**

Consider all configurations

\[ p = (p_1, \ldots, p_n) \quad p_i \in \mathbb{R}^d \]
\[ p \in \mathbb{R}^d \]

The rigid congruences act on \( \mathbb{R}^d \)

\[ (p_1, \ldots, p_n) \mapsto (Tp_1, \ldots, Tp_n) \]

where \( T \) is a translation

\[ T = A + b \quad A^T A = I \quad b \text{ translation orthogonal} \]

\[ p \mapsto Tp \]

The orbit of the configuration \( p \in \mathbb{R}^d \) is

\[ O_p = \{ \alpha p : \alpha \in \mathbb{R} \} \]

The affine span of \( O_p \) is all of \( \mathbb{R}^d \).

For "most" configurations \( p \), \( O_p \) is a manifold.

Dimension of manifold is \( \frac{d(d+1)}{2} \), when span is \( d^2 d^2 \).

We want to look at tangent space to \( p \).
Lie groups?

Let \( T_p \) be the tangent space of \( G \) at \( p \). Tangent space is the vector space of all infinitesimal elements of \( G \) at \( p \).

What does orthogonal complement look like?

We think now in terms of "forces" on the configuration \( p \). We say that a force \( F = (F_1, \ldots, F_n) \), \( F_i \in \mathbb{E} \), is an equilibrium force if \( F \) is orthogonal to \( T_p \) the tangent space of \( G \) at \( p \) at \( p \).

How do you calculate when a force \( F \) is an equilibrium force?

The definition says that \( F \cdot p^1 = 0 \) for all \( p^1 \in \mathbb{E} \) for all vectors in the tangent space.

So \( \sum_{i=1}^{n} F_i \cdot p_i^1 = 0 \) for all trivial infinitesimal motions.

Translational part: In particular when \( p_i^1 = b = \text{constant} \in \mathbb{E} \)

\[ \sum_{i=1}^{n} F_i \cdot b = 0 \] for all \( b \).

So we get that \( \sum_{i=1}^{n} F_i = 0 \) (linear momentum is 0).

Orthogonal part: This says that for every skew-symmetric matrix \( S = -S^T \)

\[ \sum_{i=0}^{n} F_i \cdot S p_i = 0. \]
\[ \sum F_i = 0 \]

Angular momentum \( \neq 0 \).

Angular momentum \( = 0 \)

Push on it & nothing happens.

Side Bar: Exterior Algebra (homogeneous version of exterior calculus).

Let \( e_{i_1 \ldots i_d} \) be a "standard" basis for \( \mathbb{R}^d \).

Define for any \( 1 \leq k \leq d \), define symbols

\[ \omega = e_{i_1 \ldots i_k} \ldots e_{i_d} \]

And we look at equivalence classes where \( e_i \cdot e_j = -e_j \cdot e_i \) and associativity holds.

The vector space generated by these equivalence classes is called \( \Lambda^k \).

That's a capital \( \Lambda \).

Note that a basis for \( \Lambda^k \) is the set \( e_{i_1 \ldots i_k} \) where \( i_1 < i_2 < \ldots < i_k \).

So the dimension of \( \Lambda^k \) is \( \binom{d}{k} \).
Note that \( \Lambda \wedge \omega = 0 \quad \forall \omega \in \Lambda^k \).

Also, \( \Lambda^k \wedge (\omega_2 + \omega_3) = \Lambda^k \omega_2 + \Lambda^k \omega_3 \).

\( \Lambda^d \otimes \mathbb{I}^d \).

\( \Lambda^{d-1} \otimes \mathbb{I}^d \) for instance \( e_1 \Lambda \ldots e_k, \ldots, e_1 \Lambda \ldots e_d \rightarrow (-1)^{k+1} e_k \).

\( \Lambda^d \otimes \mathbb{I}^d \) (\( p_1 \Lambda \ldots p_{d-1} = \det [p_1 \ldots p_{d-1}] \)).

When \( d = 3 \) \( p_1 \Lambda p_2 \leftrightarrow p_1 \times p_2 \),

Check:

\[
\begin{align*}
    e_1 \Lambda e_3 &= e_3 \\
    e_2 \Lambda e_3 &= e_1 \\
    e_1 \Lambda e_2 &= -e_3 \Lambda e_1 \\
    e_3 \Lambda e_1 &= e_2
\end{align*}
\]

For any skew-symmetric matrix, there are elements \( s \in \Lambda^{d-2} \) associated with it.

Let \( p_1, p_2 \in \mathbb{I}^d \) \( (p_1, p_2) \rightarrow p_1 \Lambda s \Lambda p_2 \), is a bilinear form.

Let \( s \in \Lambda^{d-2} \) \( (p_1, p_2) \rightarrow p_1 \Lambda s \Lambda p_2 \), is \( \Lambda^d \otimes \mathbb{I}^d = \mathbb{I}^d \) is also a bilinear form.

Note both forms are skew-symmetric:

\[
\begin{align*}
    p_1 \Gamma S p_2 &= -p_2 \Gamma S p_1 \\
    p_1 \Lambda s \Lambda p_2 &= -p_2 \Lambda s \Lambda p_1
\end{align*}
\]
So we can associate $S \leftrightarrow S = \text{skew matrix}$

With this in mind,

$$\sum_{i=1}^{n} F_i S \pi_i = 0$$

Now the condition for being in equilibrium is

$$\sum_{i=1}^{n} F_i \Lambda \pi_i = 0 = \sum_{i=1}^{n} F_i \Lambda \pi_i \Lambda S = \sum_{i=1}^{n} F_i \Lambda \pi_i \Lambda S$$

For all $s \in \Lambda^d$. Thus we must have

$$\sum_{i=1}^{n} F_i \Lambda \pi_i = 0$$

What is a form? $\sum_{i=1}^{n} F_i = 0$

Here we have forms getting that look like

$$\pi_1 \Lambda \cdots \Lambda \pi_k \in \Lambda^k$$

We claim $\Lambda^k$ is a vector space.

How do we add? How do we multiply by scalar?

$$\pi_1 \Lambda \cdots \Lambda \pi_k + q_1 \Lambda \cdots \Lambda \pi_k = (\pi_1 + q_1) \Lambda \cdots \Lambda (\pi_k + q_k)$$

and we have $e_i \Lambda \Lambda e_j = -e_j \Lambda \Lambda e_i$

So $e_i \Lambda \Lambda e_i = 0$

Suppose $d = 3$, $k = 2$

we have $e_i \Lambda \Lambda e_j$ where $i,j \in \{1,2,3\}$

basis: $e_1 \Lambda e_2, e_2 \Lambda e_3, e_1 \Lambda e_3$
homework stuff

graph

not infinitesimal rigid

4 cases with 6 vertices

\[ \text{degree 3} \]

---

Question: Let \( P \) be a convex polytope in \( \mathbb{E}^3 \), where each facet is a triangle. Can you color the edges of \( P \) with 3 colors such that each facet has 3 different colors for its edges (follows from the 4 color theorem). 4 color vertices from \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) \((a, b), (c, d)\) at 2 vertices \((a-c, b-d)\) on edge. Then at other vertex would need \((a, b)\) to trace same label of other edge. So would work.

Equilibrium forces for a configuration

\[ F = \{ F_1, \ldots, F_n \} \]

\[ P = \{ p_1, \ldots, p_n \} \]

\[ \sum F_i = 0 \quad \text{and} \quad \sum_{i=1}^n F_i \cdot A_{p_i} = 0 \]

\[ \text{Exterior complement of } T_p \quad \text{all sits in } \mathbb{E}^{3n} \]

configuration space
Example: For \( p = (p_1, \ldots, p_n) \), define
\[
F_{ij} = \begin{cases}
0 & \text{if } p_i = p_j = p_k = 0 \\
 p_i - p_j & \text{if } p_i - p_j + p_j - p_k = 0
\end{cases}
\]

Let \( \mathcal{E} \) be the equilibrium forces at \( p \).
\[\dim \mathcal{E} = nd - \frac{d(d+1)}{2}\]
when the affine span of \( p \) is \( 1\mathbb{E}^d \) (or \( d-1 \) dimensional).

Notice also that the rigidity matrix
\[ R(p) = \begin{bmatrix}
1 \\
F_{ij}
\end{bmatrix} \]
for \( \mathbf{f} = (F_1, \ldots, F_n) \).

Let \( \mathbf{f} \) be any equilibrium force for a configuration \( p \).
We say \( \mathbf{f} \) can be resolved by a stress \( \mathbf{\sigma} \) for \( \mathcal{G}(p) \) if for each \( \mathbf{c} \)
\[
F_i = \sum_{j} \omega_{ij} (p_j - p_i)
\]

Example:

But this cannot be resolved:
We say a (tensegrity) framework is statically rigid if every equilibrium force can be resolved by a proper stress \( \sigma \).

Result that we will hopefully get to someday:
static rigidity & infinitesimal rigidity are equivalent.

Note that the only possible \( F \)'s that can be resolved are equilibrium forces.

\[
\sigma R(p) = \left[ \sum \sigma_{ij} (p_i - p_j) \right] = \sum \sigma_{ij} F_{ij}
\]

Each \( F_{ij} \) is an equilibrium force so any linear combo is an equilibrium force as well.

In fact \( \sigma R(p) \) proper is the set of resolvable forces.

Suppose \( G(p) \) is an inf. rigid bar framework.
Let \( p' \) be an inf. flex of \( G(p) \).
So \( R(p) p' = 0 \).
Let \( \tilde{p}' \) be the orthogonal projections of \( p' \) into \( E \) the space of equilibrium forces.
Claim: \( \tilde{p}' \) is also an inf flex of \( G(p) \) and \( p' = 0 \)
if and only if \( p' \) is trivial.

\[ \tilde{p}' = p' - u, \quad u \text{ is trivial.} \]

When \( i+j \) = member of \( G \),
\[ (p_i - p_j) \cdot (\tilde{p}' - \tilde{p}) = (R_i - p_j) \cdot (p_i' - p_j') = (R_i - p_j) \cdot (p_i' - p_j') = 0 \]

So \( \bar{p}' \) is an inf. flex of \( G(p) \) as well.

So if \( G(p) \) is statically rigid, then there is a stress \( \omega \) where \( \omega \) resolves \( \bar{p}' \in \mathcal{E} \). So \( \omega R(p) = \bar{p}' \Rightarrow \omega R(p) \bar{p}' = \bar{p}' \bar{p} \Rightarrow \bar{p}' = 0 \) then \( \bar{p}' = 0 \).

Thus if \( G(p) \) is not inf. rigid, then it is not statically rigid.

Say affine span of \( p \) is all of \( \mathbb{R}^d \).
So \( G(p) \) as a bar framework is inf. rigid iff 
\[
\text{rank } R(p) = nd - \frac{d(d+1)}{2}
\]

and it is statically rigid iff the image of
\[
\omega \mapsto \omega R(p)
\]
is all of \( \mathcal{E} \).

but \( \dim \mathcal{E} = nd - \frac{d(d+1)}{2} \), if \( \text{rank } R(p) = nd - \frac{d(d+1)}{2} \)

So if \( G(p) \) bar framework, then
inf. rigidity \( \iff \) static rigidity.

Would like to extend this to tensegrity frameworks.
Static rigidity $\iff$ Forces (Engineering viewpoint)
Infiniteesimal rigidity $\iff$ Velocities (Differential Geometry)
Herglotz $\iff$ Cauchy theorem

We want to show that static rigidity and infiniteesimal rigidity are equivalent for tensegrities. This will use duality.

Static rigidity means every equilibrium force $F$ can be resolved by a proper equilibrium stress, i.e., $R(p) = F$, a proper

Infiniteesimal rigidity means all infiniteesimal flexions $G(p)$ are trivial, i.e., $R(p) p' \leq p = 0 \Rightarrow p' \text{ trivial}.

Convexity Results:
1. Let $X$ = closed convex set in $\mathbb{R}^d$ and $p \in \mathbb{R}^d - X$.

Then there is a unique point $q$ in $X$ nearest to $p$.

2. There is a half space containing $X$, but not $p$.

3. If $X$ is a cone from the origin, then $0 \in \partial X$.

Let $E$ be a linear subspace of $\mathbb{R}^d$. Let $a_1, \ldots, a_n \in E$.
Define $H_{a_i} = \{ p \in E | p \cdot a_i \leq 0 \}$
Then $\bigcap_{i=1}^n H_{a_i} = \{ 0 \}$ if and only if for all $b \in E$ there are scalars $\lambda_1, \ldots, \lambda_n$ such that $\sum_{i=1}^n \lambda_i a_i = b$. ($\star$)
Proof:

(\Leftarrow) Suppose for all \( b \) there are scalars \( \lambda_i \) such that (*) holds. Let \( p \in \bigcap H_{a_i} \). Write \( p = \sum_{i=1}^{n} \lambda_i a_i \); each \( \lambda_i \geq 0 \). Then

\[
p \cdot p = p \cdot (\sum_{i=1}^{n} \lambda_i a_i) = \sum_{i=1}^{n} \lambda_i p \cdot a_i \leq 0
\]

\[\Rightarrow \quad p = 0\]

(\Rightarrow) Suppose \( \bigcap_{i=1}^{n} H_{a_i} = \{0\} \).

Let \( b \in \mathcal{E} \). Let \( \mathcal{X} = \{ p \mid p = \sum_{i=1}^{n} \lambda_i a_i \} \) be convex and including 0. Let \( b \notin \mathcal{X} \). Then there exists a hyperplane \( H_b \) that separates \( \mathcal{X} \) from \( b \). Let \( H_b = \{ p \mid p \cdot v \leq 0 \} \) for some \( v \in H_{a_i} \).

Since \( b \notin \mathcal{X} \), \( v \in \mathcal{X} \). So \( b \in H_b \). Thus \( b \in \mathcal{X} \).

\[\blacksquare\]

? Can you make a cube rigid with 7 cables?

\[\blacksquare\]

(need to put some cables inside)

Theorem: Infinitesimal rigidity + static rigidity are equivalent.

Proof: Let \( G(p) \) be infinitesimally rigid, and let \( F \) be an equilibrium force on the configuration \( p = (p_1, \ldots, p_n) \).

Let \( \mathcal{E} = \) equilibrium forces for \( p \). So we are looking for a stress \( \mathcal{W} \) such that \( \mathcal{W} R(p) = F \) and each \( \mathcal{W} \) is proper.

Let \( R_+(p) = R(p) \) with the first rows multiplied by \(-1\). So there is a proper stress \( \mathcal{W} \) if and only if there is a stress \( \mathcal{W} \geq 0 \) such that \( \mathcal{W} R_+(p) = F \).
The condition for infinitesimal rigidity of $F(p)$ is $R(p) p^1 \leq 0$ implies $p^1 = 0$. If $p^1 \in \mathbb{R}$, let $q_1 = c^1 \cdot p^1 \cdot \text{row}_q R(p)
$ This says $\cap H_0 = \emptyset$. 

So our duality result implies that inf. rigidity

$\iff$ static rigidity.
Theorem (1980 - Roth - Whiteley)
Let $G(p)$ be a tensegrity framework in $\mathbb{R}^d$. Let $G(p)$ be the underlying bar framework.
Then $G(p)$ is infinitesimally rigid ($\iff$ statically rigid)
$\iff$ the following two conditions hold:
1) $G(p)$ is infinitesimally rigid and
2) $G(p)$ has a proper stress such that $w_{ij} \neq 0$ for each cable or strut $\hat{e}_{i,j}$.
(proper means $w_{ij}$ has right sign depending on whether $\hat{e}_{i,j}$ is a cable or strut)

Proof:
($\iff$) We will begin by showing (i) $\implies$ (ii) $\iff$ $G(p)$ is rigid.
(supposedly this is the "easy" way).
Suppose (i) and (ii) hold. We want to show $G(p)$ is rigid. Let $p'$ be an inf. flex of $G(p)$.
We will show that $p'$ is trivial.
Let $w$ be the proper equilibrium stress given by (ii).
Since $w$ is an equilibrium stress, $w^TR(p)w = 0$ where $R(p)$ is rigidity matrix.
So $0 = w^TR(p)p' = \sum_i w_{ij}(p_i - p_j)\cdot(p'_i - p'_j)$.
For $\hat{e}_{i,j} = \text{cable}$, $w_{ij} > 0$ and $(p_i - p_j)\cdot(p'_i - p'_j) \neq 0$.
with strict inequality unless $(p_i - p_j)\cdot(p'_i - p'_j) = 0$.
Similarly for $\hat{e}_{i,j} = \text{strut}$, $(p_i - p_j)\cdot(p'_i - p'_j) \leq 0$.
So $w_{ij}(p_i - p_j)\cdot(p'_i - p'_j) \leq 0$ with strict inequality unless $(p_i - p_j)\cdot(p'_i - p'_j) = 0$.
For $\hat{e}_{i,j} = \text{bar}$, $w_{ij} < 0$ and $(p_i - p_j)\cdot(p'_i - p'_j) \geq 0$.
So $w_{ij}(p_i - p_j)\cdot(p'_i - p'_j) \leq 0$ with strict inequality unless $(p_i - p_j)\cdot(p'_i - p'_j) = 0$.

For a bar, $\hat{e}_{i,j} = \text{bar}$, $w_{ij}(p_i - p_j)\cdot(p'_i - p'_j) = 0$. 
So $\omega R'(p)p' \leq 0$ with strict inequality unless $(p_i - p_j) \cdot (p_i' - p_j') = 0$ for all members $\xi_{i,j,3}$. So $(p_i - p_j) \cdot (p_i' - p_j') = 0$ for all members $\xi_{i,j,3}$.

So $p'$ is an inf. flex of $G(p)$. So by (i) $p'$ is trivial.

(\Rightarrow) This is the harder, supposedly more interesting, direction.

Suppose $G(p)$ is statically rigid in $E^d$. We want to show (i) and (ii) hold. (Note that we're using the fact that inf. rigidity $\iff$ static rigidity.)

Clearly (i) holds. If $G(p)$ is inf. rigid, then $G(p)$ is inf. rigid, so we just need to show (ii).

Let $F(\xi_{i,j,3}) = [0, \ldots, p_i - p_j, 0, \ldots, p_j - p_i, 0, \ldots]$

Claim: this is an equilibrium force.

Because $\sum F_k(\xi_{i,j,3}) = p_i - p_j + p_j - p_i = 0$.

$$\sum F_k(\xi_{i,j,3}) A_{p_k} = (p_i - p_j) A_{p_i} + (p_j - p_i) A_{p_j} = -p_j A_{p_i} - p_i A_{p_j} = 0$$

Let $\bar{w}(\xi_{i,j,3})$ be the resolving stress for this equilibrium force.

Now let $w_{i,j,3}$ be any $\omega w_{i,j,3}$ and $w_{i,j,3} = \bar{w}(\xi_{i,j,3})$.

This is an equilibrium stress for $G(p)$, and it is proper if $w_{i,j,3} > 0$.

Similarly for a strain $\xi_{i,j,3}$ define $w(\xi_{i,j,3})$ so that it is a proper equilibrium force stress for $G(p)$ and $w(\xi_{i,j,3}) < 0$. Then
\[ \omega = \sum_{ij} \omega(i,j) \] is a proper equilibrium.

Stress \( \sigma_{ij} \neq 0 \) for all cables + struts.

Note: Cannot ensure that bars have stress.

Example: \( \Delta \) statically rigid, but no stress.

Examples:
1. Statically rigid because

   ![Diagram](image1.png)

   \( \infty \cdot \text{rigid} \)

   \( \because \) there is a stress

2. In more generality, any Cauchy polygon is

   statically rigid:

   \( \subseteq \) \( e-s, b-s, s-c, c-b, s-b, b-c, b-b \)

3. Grünbaum polygons

   ![Diagram](image2.png)

   \( \infty \cdot \text{rigid} \)

   \( \because \) there is a stress

4. Not statically rigid regular
   no stress when hexagon not regular
   not \( \infty \cdot \text{rigid} \) when hexagon regular

That's what mathematics is about - handwaving.
Pre-stress rigidity

energy

start with \( g(p) \) a tensoenergy \{cables, bars, struts\} : \( G \)

configuration: \( p \)

We define an energy functional for each member \( E_{ij} \)

\[ l = |p_i - p_j| \]

bars:

\[ E_{ij} \uparrow \text{ unique min at } l^2 = |p_i - p_j|^2 \]

\[ \uparrow |p_i - p_j|^2 \quad \text{wait } E_{ij}'' > 0 \]

It costs us energy to move away from \( l^2 = |p_i - p_j|^2 \).

cable:

\[ \uparrow \text{ must be monotone increasing } \]

\[ \uparrow |p_i - p_j| \quad \text{wait } E_{ij}'' > 0 \]

strut:

\[ \uparrow \text{ must be monotone decreasing } \]

\[ \uparrow |p_i - p_j| \quad \text{wait } E_{ij}'' > 0 \]

For any other configuration \( q \), (that is, a configuration other than \( p \))

\[ E(q) = \sum_{i < j} E_{ij} (|q_i - q_j|^2) \]

What we had previously was a special case.
Proposition: Principle of least work applies: If \( E \) is such that \( E(p) \) is a unique minimum up to congruences of \( p \), then \( G(p) \) is rigid in \( 1E \).

Proof: Establish \( \varepsilon \) up to be the neighborhood of \( p \) in \( 1E \) such that \( B_E(p) \) is a minimum up to congruences of \( p \). Suppose \( g \in \varepsilon \) up to \( G(g) \leq G(p) \) (i.e., \( g \) satisfies the member constraints).

For cable \( l_{i-j} \), if \( |p_i - p_j| \) then \( E_{ij} \left( \frac{1}{2} (p_i - p_j)^2 \right) \leq E_{ij} \left( \frac{1}{2} (p_i - p_j^1)^2 \right) \) same for bars + struts.

So \( E(\varepsilon) \leq E(p) \). So \( g \) congruent to \( p \).

\( G(p) \) is rigid.

Def: \( G(p) \) is pre-stress stable if there are energy functions of this sort such that \( E \) is a relative min at \( p \), modulo congruences, because of "the second derivative test".

First derivative test (for minima):
Let us take the directional minima in the direction \( p' \).
We consider \( E(p + tp') \) and calculate \( \frac{dE}{dt} \) of \( E(p + tp') \) at \( t = 0 \).

\[
\frac{d}{dt} E(p + tp') = \sum_{ij} \frac{d}{dt} E_{ij} \left( \frac{1}{2} |p_i - p_j + t(p_i' - p_j')|^2 \right)
\]

\[
= \sum_{ij} E_{ij} \left( \frac{1}{2} |p_i - p_j + t(p_i' - p_j')|^2 \right) \cdot (p_i - p_j - p_i' + p_j') \cdot t + 2t (p_i - p_j') \cdot t
\]

\[
\frac{d}{dt} E_{ij} \left( \frac{1}{2} |p_i - p_j|^2 \right) = E_{ij} \left( \frac{1}{2} |p_i - p_j + t(p_i' - p_j')|^2 \right) \cdot (p_i - p_j - p_i' + p_j') + 2t (p_i - p_j') \cdot t
\]
At \( t=0 \), \( \frac{d}{dt} E(p+t p') \big|_{t=0} = 2 \sum_{i<j} E_{ij} (1(p_i-p_j) \cdot (p_i-p_j) \cdot \cdot p_i-p_j) \). 

Define \( w_{ij} = E_{ij} (P_i-P_j)^2 \).

Call \( w = (\ldots, w_{ij}, \ldots) \) the "stress".

If \( p \) is a critical point for \( E \), then
\[ \sum_{i<j} w_{ij} (p_i-p_j) \cdot (p_i-p_j) = 0 \quad \text{for all directions } p'. \]

In other words, \( w R(p) = 0 \) for all \( p' \).

So \( w R(p) = 0 \). So \( w \) is an equilibrium stress for \( G(p) \).

Note also, \( w \) is proper.

The second-derivative test.

\( p \) will be a local min if \( p \) is a critical point and
\[ \frac{d^2 E(p+t p')}{dt^2} \big|_{t=0} > 0 \quad \text{for all } p', \text{ and } \]

equality only when \( p' \) is a trivial infinitesimal flex.

\[ \frac{d^2 E(p+t p')}{dt^2} \]

\[ \frac{d^2 E_{ij}}{dt^2} \big(1(p_i-p_j) + t(p_i'-p_j')\big)^2 \]

\[ g \cdot 4 E_{ij}'' (1(p_i-p_j) + t(p_i'-p_j')^2)(p_i-p_j) \cdot (p_i'-p_j')^2 + 2 t (p_i'-p_j') \]

\[ + 4E_{ij}' (1(p_i-p_j) + t(p_i'-p_j')^2)(p_i'-p_j')^2 \]

At \( t=0 \), we have
\[ 4 E_{ij}'' (1(p_i-p_j)^2) (p_i-p_j) (p_i'-p_j')^2 + 4 w_{ij} (p_i'-p_j')^2 \]
\[
\frac{d^2 E(p)}{dt^2} = 4 \sum_{i<j} \omega_{ij} |p_i - p_j|^2 + 4 \sum_{i<j} c_{ij} (p_i - p_j) \cdot (p_i - p_j)'^2
\]

where \( c_{ij} = E_{ij}(1|p_i - p_j|^2) > 0 \)

Call \( c_{ij} \) 's **stiffness coefficients**

Recall \( (p_i)' \otimes I^d \cdot p_i = \sum_{i,j} \omega_{ij} (p_i - p_j)'^2 \)

Let \( C = \begin{bmatrix} \ddots & 0 \\ 0 & c_{ij} \\ \end{bmatrix} \) = **stiffness & diagonal matrix**

Calculate \( [R(p) \cdot p_i]'^T C [R(p) \cdot p_i]' = R(p) \cdot p_i \cdot \begin{bmatrix} \vdots \\ c_{ij} (p_i - p_j) \cdot (p_i - p_j)' \end{bmatrix} \)

\[
= \sum_{i<j} c_{ij} \left[ (p_i - p_j) \cdot (p_i - p_j)' \right]^2
\]

\[
= (p_i)'^T R(p)' \cdot C \cdot R(p) \cdot p_i
\]

**Symmetric**

**Positive definite matrix**

Second derivative is \( R \otimes I^d + R(p)' C R(p) \)
April 19

Today: Finish "pre-stressability"
Start Maxwell-Crame correspondence

Reference for pre-stressability - paper of Connelly & Whiteley
for Maxwell-Cramen - papers by Croce & Whiteley, Gluck (1973)

"Almost all..."

Recall that \( G(p) \) is pre-stress rigid or (pre-stressable) if there is a proper equilibrium stress for \( G(p) \) such that \( \Omega \otimes I^d + (R(p))^TR(p) \) is positive semi-definite with kernel exactly the trivial infinitesimal frame of \( G(p) \), where \( \Omega \) is the stress matrix associated with \( R(p) \), and \( R(p) \) is the rigidity matrix for the corresponding bar framework \( G(p) \).

Example:

\[ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ has one negative eigenvalue} \]

but \( (R(p)) + (R(p))^TR(p) \) is positive semi-definite of max rank.

This is unstable for large proper stress but stable for a small proper stress, since in that case the \( (R(p))^TR(p) \) will dominate.

Test for pre-stress stability.
Let \( K_p = \{ p' \in \text{End} | (R(p))p' = 0 \} \).

Consider the quadratic form associated to the stress energy \( \Omega \otimes I^d \) restricted to \( K_p \).
\[ \langle (p')^T \Omega \otimes I^d \rangle p' \]
Proposition: \( G(p) \) is pre-stress stable if \( (\mathcal{L} \otimes I^p)^T \mathcal{L} \mathcal{P} \) is positive definite on \( K_p \Lambda \mathcal{E} \) where \( \mathcal{E} = \) equilibrium forces.

Proposition: If \( G(p) \) is \( \infty \)-inf. rigid, it is pre-stress stable.

Proof:

If \( G(p) \) is a bar framework, choose \( w = 0 \), then \( \mathcal{P}(p)^T \mathcal{P}(p) \geq 0 \) and has only the trivial kernel.

If \( G(p) \) is a tensegrity framework, choose a stress that is non-zero on all the cables + struts, by Roth-Whiteley (Theorem that \( \infty \)-inf. tensegrity has non-zero stress) then choose \( \varepsilon \geq 0 \) s.t. \( \varepsilon \mathcal{E} \mathcal{P}(p)^T \mathcal{P}(p) \mathcal{E} \) is still positive definite on \( \mathcal{E} = (\text{trivials}) \) and so \( \varepsilon \mathcal{E} \mathcal{P}(p)^T \mathcal{P}(p) \mathcal{E} \) serves as a stabilizing stress for pre-stress stability.

Proposition: If \( G(p) \) is pre-stress stable, then it is rigid.

Proof:

So in \( \mathcal{E} \mathcal{P} \), \( \mathcal{E}(q) = \sum_{i,j} E_{ij}(q_i - q_j) \)

\( \mathcal{E} \mathcal{P} \) has a strict local minimum.

Principle of least work applied. So if \( \mathcal{E}(q) < \mathcal{E}(p) \), then \( \mathcal{E}(q) \leq \mathcal{E}(p) \), \( (q - p) \leq 0 \), then this contradicts local minimum. So \( \mathcal{E}(q) = \mathcal{E}(p) \).

So this is an alternate way to show \( \infty \)-inf. rigid \( \Rightarrow \) rigid.

This is the basic underlying geometry to engineering. Can you think of something that is rigid but not pre-stress stable?
rigid but not prestress stable
The correspondence from lift to eq. stress is natural. Under specific certain assumptions, you can go from eq. stress to a lift.

What are the objects that are concerned here?
What is the correspondence?

Inference motion of the planar surface in $E^2$
We'll consider triangulated surfaces.
And we'll allow self-intersections.

Infinitesimal motions $\Rightarrow$ Equilibrium Stress

We have already seen duality between infinitesimal strains.

**The Objects**

We eventually come to bar frameworks in $\mathbb{R}^3$.

**Definition:** A simplex (closed) is $\sigma = \langle p_0, \ldots, p_n \rangle \subset \mathbb{R}^N$.

$\dim \sigma = n \land p_0, \ldots, p_n$ are affine independent.

For us, $\sigma^0 = \text{point}$.
$\sigma^1 = \text{line segment}$.
$\sigma^2 = \text{triangle (filled in)}$.

A face of a simplex $\sigma^n$ is $\langle p_{i_1}, \ldots, p_{i_k} \rangle$.

A simplex has $2^n + 1$ faces (counting the empty set).

A simplicial complex $K = \{\sigma\}$ collection of simplices such that

1. If $\sigma \in K$, all faces of $\sigma$ are in $K$.
2. If $\sigma, \tau \in K$, $\sigma \cup \tau$ is a face of $\sigma + \tau$.
3. $\emptyset \in K$.

For a simplicial complex $K$,

$K^{(i)} = \{ \sigma \in K \mid \dim \sigma \leq i \}$

For $\sigma \in K$, star of $\sigma$ is $\text{star}(\sigma, K) = \{ \tau \in K : \tau \supseteq \sigma \}$.

This is not a complex.
The closed star of $a$ in $K$ is
\[ \text{st}(a, K) = \{ x \mid x \text{ is a face of } st(a, K) \} \]

The link of $a$ in $K$ is
\[ \text{lk}(a, K) = \text{st}(a, K) - \text{st}(a, K) \]

A 0-sphere is a simplicial complex with just 2 vertices and $\emptyset$.

A 1-sphere is a cycle of 1-simplices and their faces. Example:
\[ S^1 \]

A combinatorial 2-manifold is a 2-dimensional simplicial complex $K$ where for each vertex $v_i$,
\[ \text{lk}(v_i, K) = 1 \text{-sphere} \]

Examples:
1. boundary of 3-simplex
2. boundary of octahedron
3. boundary of regular icosahedron
   \[ \text{lk}(v_1, K) = \]
A 2-sphere is an orientable 2-manifold with $X = 2$.

For a simplex $\sigma \in \mathbb{E}^N$, an affine map $f: \sigma \to \mathbb{E}^d$ is the restriction of an affine linear map from $\mathbb{E}^N \to \mathbb{E}^d$.

Note: If $f$ is one-to-one, then $f(\sigma)$ is a simplex in $\mathbb{E}^d$.

Let the underlying space of $K$ be $\bigcup_{\sigma \in K} \sigma \subseteq \mathbb{E}^N$.

A singular complex is a map $f: K \to \mathbb{E}^d$ such that for every simplex $\sigma \in K$, $f|_{\sigma}$ is affine linear.

Example: $\square \xrightarrow{f} \not\Delta$, not a complex.
A singular complex is non-degenerate if \( f/\delta \) is one-to-one.

We consider singular, non-degenerate orientable 2-manifolds in \( \mathbb{R}^3 \).

For any such singular, non-degenerate 2-manifold \( \mathcal{F} \), let \( p = (p_1, \ldots, p_n) \) be the image of the vertices of \( \mathcal{F} \). Let the 1-skeleton correspond to the bars of \( \mathcal{F} \) framework \( G(p) \) in \( \mathbb{R}^3 \).

Now let \( p' \) be an infinite flex of \( G(p) \) in \( \mathbb{R}^3 \).

Call \( \text{IF} = \sum p' = (p'_1, \ldots, p'_n) \) if \( p' \) is an infinite flex of \( G(p) \).

Let \( \mathbf{d} = \sum \omega = (\omega_1, \ldots, \omega_n) = \omega R(p) = 0 \).

The Maxwell-Cromwell correspondence is a function from \( \text{IF} \) to \( \mathbb{R}^3 \) such that \( \ker MC = \) trivial infinite flexes of the singular complex.

For each \( \mathbb{R} \) 1-simplex \( e' \leftrightarrow e_{ij}3 \) of \( G(p) \), let \( \Theta_{ij} \) = oriented dihedral angle for the edge \( e_{ij}3 \).

Corresponding to any \( p' \) is \( \Theta_{ij} \).

Then the stress on \( e_{ij}3 \) is

\[ w_{ij} = \Theta_{ij} / |p_i - p_j| \]
Notice that: 

\[ w_j > 0 \quad \text{if} \quad \theta_j > 0 \]

\[ w_j < 0 \quad \text{if} \quad \theta_j < 0 \]

\[ w_j = 0 \quad \text{if} \quad \theta_j = 0 \]
April 26

stiff on

Homework:

\[ M^2 \text{ in } E^3 \]
\[ f : K^2 \to E^3 \]

pretend this

circle tangent
to all the
circles

Draw line between

each of the circles

that are tangent.

bi-pyramid

In this case, \( M^2 \) is a singular 2-sphere.

We have \( p = (p_1, \ldots, p_n) \) is the 0-skeleton of our
possibly singular, non-degenerate manifold \( M^2 \).

Let \( p' \) be an infinitesimal flex of \( G(p) \), the 1-skeleton of \( M \).

Let \( \theta_{ij} \) be the dihedral angle associated to
\( e_i \) and \( e_j \). The infinitesimal \( p \) defines a corresponding \( \theta_{ij}' \).

\[ p 
\mapsto p + p' \]
defines \( \theta_{ij}'(\epsilon) \), then \( \theta_{ij} = \left. \frac{d}{d\epsilon} \theta_{ij}'(\epsilon) \right|_{\epsilon=0} \).
So we define \( w_{ij} = \frac{Q_{ij}}{|p_i - p_j|} \)

**Claim:** \( w = (\ldots, w_{ij}, \ldots) \) is an equilibrium stress for \( G(p) \).

**Proof:** (of claim)

This proof has to do with beginning part of Lie Algebras.

Look at any vertex \( p_0 \) of \( G \).

Let \( p_1, \ldots, p_n \) be the adjacent vertices in cyclic order around \( p_0 \).

Say \( p_0 = 0 \).

Let \( \psi_i = \) the angle between \( p_i \) and \( p_{i+1} \).

**Note:** \( \psi_i = 0 \).

We also will assume that \( |p_i| = 1, \quad i = 1, \ldots, n \).

Define \( A_0, \psi \) \( \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

\( A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & -\sin \psi \\ 0 & \sin \psi & \cos \psi \end{bmatrix} \)

Assume by changing by an appropriate rigid motion

\( p_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix} \)
\[ B_{\psi_1}^{-1} P_3 = P_1 \]

Then, \( A_{\psi_2}^{-1} B_{\psi_1}^{-1} \) rotates the whole configuration into a position similar to the original. Now \( A_{\psi_2} B_{\psi_1} P_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \)

and \[ A_{\psi_2}^{-1} B_{\psi_1} P_3 = \begin{bmatrix} \cos \psi_2 \\ \sin \psi_2 \\ 0 \end{bmatrix} \]

Then, since the link of a vertex is a cycle, \( A_{\psi_2}^{-1} B_{\psi_1}^{-1} A_{\psi_2} B_{\psi_1} = I \), or equivalently \( A_{\psi_1} B_{\psi_2} A_{\psi_1} B_{\psi_2} \ldots A_{\psi_n} B_{\psi_n} = I \) (at something similar).

We want to take the derivative of this. The product rule of differentiation applies to matrices. But remember \( B_{\psi_i} = 0 \).

\[ A_{\phi} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\sin \Theta & \cos \Theta & 0 \\ 0 & \cos \Theta & -\sin \Theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Theta' \]

So, taking the derivative, we get:

\[ \Delta = A_{\phi_1} B_{\psi_1} A_{\phi_2} \ldots B_{\psi_n} A_{\phi_1} B_{\psi_1} A_{\phi_2} \ldots B_{\psi_n} \ldots A_{\phi_1} A_{\phi_2} B_{\psi_n} \]

Each term \( \Delta' \) of the form \( A_{\phi_1} B_{\psi_1} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\sin \Theta & \cos \Theta & 0 \\ 0 & \cos \Theta & -\sin \Theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ldots B_{\psi_n} \Theta' \).
after lots of work, we can conclude that
\[ \sum_i \theta_i' p_i = 0 \]

then \( \theta_i = \omega_{i0} \)

That is equivalent to \( \sum_i \frac{\theta_{i0}'}{p_i - p_0} = 0 \).

\[ \omega_{ij} = \frac{\theta_{ij}'}{|p_i - p_j|} \]

(End of proof of claim)

Inf. Area of general oriented 2-manifold \( M \)

Equilibrium stress on \( M \) \( \Rightarrow \) stresses in 2-sphere

Equilibrium stress on \( M \subset \mathbb{R}^2 \) \( \leftrightarrow \) Reciprocal diagram in \( \mathbb{R}^2 \)

Stresses on \( M = S^2 \) \( \rightarrow \) Lie's of \( M \) to \( \mathbb{R}^3 \)