Tensegrities and Global Rigidity

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Abstract

A tensegrity is finite configuration of points in $\mathbb{E}^d$ suspended rigidly by inextendable cables and incompressable struts. Here it is explained how a stress-energy function, given by a symmetric stress matrix, can be used to create tensegrities that are globally rigid in the sense that the only configurations that satisfy the cable and strut constraints are congruent copies.

1 Introduction

In 1947 a young artist, Kenneth Snelson, was intrigued with a particular structure that he invented. It was a few sticks that were suspended rigidly in midair without touching each other. It seemed like a magic trick. When he showed this to the entrepreneur, builder, visionary, and self-styled mathematician, R. Buckminster Fuller, he was inspired to call it a *tensegrity* because of its "tensional integrity". Fuller talked about them and wrote about them extensively. Snelson went on to build a great variety of fascinating tensegrity sculptures all over the world including a 60 foot work of art at the Hirschhorn museum in Washington, DC. as shown in Figure 1.

Why did these tensegrities hold up? What were the geometric principles? They were often under-braced, and they seemed to need a lot of tension for their stability. So Fuller's name, tensegrity, is quite appropriate.

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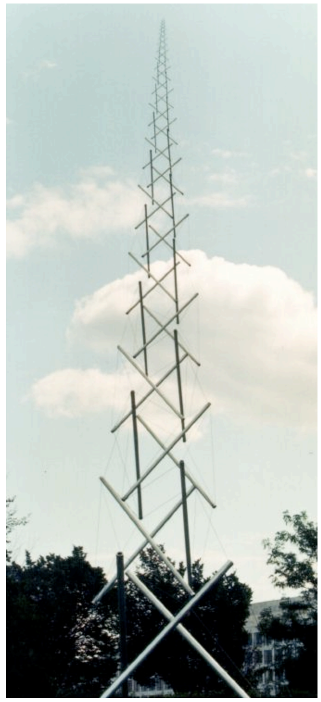


Figure 1:

My proposal is that there is a very reasonable and pleasant model to describe the stability of most of the tensegrities that Snelson and others have built. There are results that can be used to predict the stability of a tensegrity, and there is a calculation that seems to reasonably imply stability, but also to create tensegrities that are stable. In the following, up to Section 8, there will be a self -contained elementary development of a set of principles that can be used to understand many of the Snelson-like tensegrities. This relies on the properties of the stress matrix, discussed in Section 5. Then in Section 8 the properties of the stress matrix are applied to generic configurations of bar tensegrities (usually called bar frameworks), where there have been a lot of exciting new results recently, and the ideas will be outlined.

The discussion here emphasizes the stress matrix and the stress-energy functional, and largely ignores the first-order theory, about which a lot has been written.

There are several quite interesting applications of the theory of tensegrities. Of course, there is a natural application to structural engineering, where the pin-jointed bar-and-joint model is appropriate for an endless collection of structures. See [34, 1, 37] for example. See [31, 38, 35, 36, 32] for the first-order theory, and see [17] for the more general approach that combines the first-order theory and the stress matrix approach that is developed here. In computational geometry, there was the carpenter's rule conjecture, inspired by a problem in robot arm manipulation. This proposes that a non-intersecting polygonal chain in the plane can be straightened, keeping the edge lengths fixed, without creating any self-intersections. The key idea in that problem uses basic tools in the theory of (first-order) tensegrity structures. See [14, 13] as well as Subsection 7.8 here. Granular materials of hard spherical disks can be reasonably modeled as tensegrities, where all the members are struts. Again the theory of tensegrities can be applied to predict behavior and provide the mathematical basis for computer simulations as well as predict the distribution of internal stresses. See [19].

2 Notation

Formally define a tensegrity as a finite set of labeled points called *nodes*, where some pairs of the nodes are connected with inextendable *cables*, some pairs of nodes are connected with incompressible *struts*, and some pairs of nodes are connected with inextendable, incompressible *bars*. The cables, struts, and bars are all called *members* of the tensegrity. A continuous motion of the nodes, starting at the given configuration of a tensegrity, where the member constraints are satisfied, is called a *flex* of the tensegrity. Any configuration of points has continuous flexes, such as rotations, translations, and their compositions that are restrictions of congruences of the whole space. These are called *trivial flexes* of the tensegrity. If the tensegrity has only the trivial flexes that satisfy the member constraints, then it is *rigid* in $\mathbb{E}^d$. Otherwise it is called *flexible*. Note that members can cross one another, intersect, and we are not concerned with what materials one might use to build a tensegrity that enforce the member constraints. This is a purely geometric object. Figure 2 show some examples of rigid and flexible tensegrities in the plane and space.

The rigid tensegrity in space in Figure 2 is one of the original objects made by Snelson. It is quite simple but suspends three sticks, the struts, rigidly without any pair of them touching. Indeed Snelson does not like to

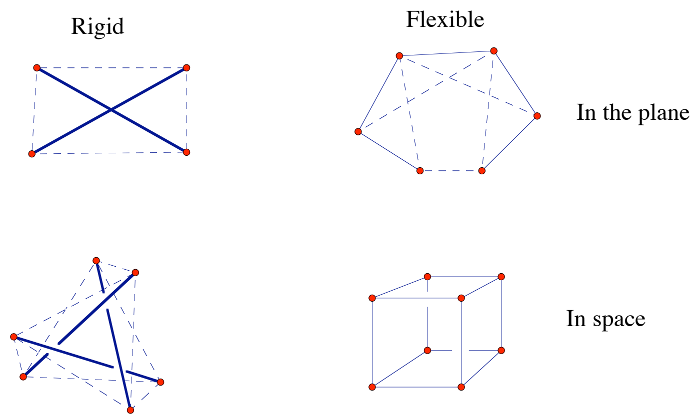


Figure 2: Nodes are denoted by small round points, cables by dashed line segments, struts by solid line segments, and bars by thin line segments.

call an object made of cables and struts a tensegrity unless all the struts are completely disjoint, even at their nodes. If a tensegrity, by the definition here, is such that the struts are disjoint, while all the other members are cables, it will be called a *pure* tensegrity.

One can build many of the rigid tensegrities shown here with rubber (or plastic) bands for cables, and dowel rods with a slot at their ends serving as struts or bars.

In what follows, there will be some discussion of techniques for computing the rigidity of tensegrties. As a by-product of this analysis global rigidity, defined in Section 3, will emerge naturally.

Let G denote the underlying tensegrity graph of nodes, where the edges of G, the members, are each labeled as cables, struts or bars. Let $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$ denote the configuration of nodes, where each $\mathbf{p}_i$ is a vector in $\mathbb{E}^d$. The whole tensegrity is denoted as $G(\mathbf{p})$. I regard this notation as somewhat bezarre, but it is helpful to distinguish between the configuration $\mathbf{p}$ and the way the pairs of nodes are connected with the three types of members. There are occasions, where the graph G is not needed, and the configuration can stand alone by itself, and there are other times when on the graph G is relevant, and the configuration is put in the background.

An important concept is the notion of a *stress* associated to a tensegrity, which is a scalar $\omega_{ij} = \omega_{ji}$ associated to each member $\{i, j\}$ of G. Call the vector $\omega = (\ldots, \omega_{ij}, \ldots)$, the *stress*. We can suppress the role of G here by simply requiring that $\omega_{ij} = 0$ for any non-member $\{i, j\}$ of G. A stress $\omega = (\ldots, \omega_{ij}, \ldots)$ is *proper* if $\omega_{ij} \geq 0$ for a cable $\{i, j\}$ and $\omega_{ij} \leq 0$ for a strut $\{i, j\}$. There is no condition when $\{i, j\}$ is a bar. We say a proper stress ω is *strict* if each $\omega_{ij} \neq 0$ when $\{i, j\}$ is a cable or strut. One should be careful here, since in the paper [32], a proper stress is called what is strict and proper here. I prefer the definition here since it is convenient to not necessarily insist that all proper stresses are strict. One should also be careful not to confuse the notion of stress here with that used in structure analysis, in physics or in engineering. There stress is defined as a force per cross-sectional area. In the set-up here there are no cross-sections; the scalar ω_{ij} is better interpreted as a force per unit length.

Let $\omega = (\ldots, \omega_{ij}, \ldots)$ be a proper stress for a tensegrity graph G. For any configuration $\mathbf{p}$ of nodes in $\mathbb{E}^d$, define the *stress-energy* associated to ω as

$$E_\omega(\mathbf{p}) = \sum_{i<j} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j)^2, \tag{1}$$

where the product of vectors is the ordinary dot product, and the square of a vector is the square of its Euclidean length.

A conceit I like is to say the tensegrity $G(\mathbf{p})$ *dominates* the tensegrity $G(\mathbf{q})$, and write $G(\mathbf{q}) \leq G(\mathbf{p})$, for two configurations $\mathbf{q}$ and $\mathbf{p}$, if

$$\begin{array}{rcll} |\mathbf{p}_i - \mathbf{p}_j| & \geq & |\mathbf{q}_i - \mathbf{q}_j| & \text{for } \{i, j\} \text{ a cable,} \\ |\mathbf{p}_i - \mathbf{p}_j| & \leq & |\mathbf{q}_i - \mathbf{q}_j| & \text{for } \{i, j\} \text{ a strut and} \\ |\mathbf{p}_i - \mathbf{p}_j| & = & |\mathbf{q}_i - \mathbf{q}_j| & \text{for } \{i, j\} \text{ a bar.} \end{array} \tag{2}$$

So if $G(\mathbf{p})$ dominates $G(\mathbf{q})$ and ω is a proper stress for G, then $E_\omega(\mathbf{p}) \geq E_\omega(\mathbf{q})$, and when ω is strict and $E_\omega(\mathbf{p}) = E_\omega(\mathbf{q})$, then $|\mathbf{p}_i - \mathbf{p}_j| = |\mathbf{q}_i - \mathbf{q}_j|$ for all the members $\{i, j\}$ of G. The conditions of (2) are called the *tensegrity constraints*.

3 Local and global rigidity

So, more formally, a tensegrity $G(p)$ is rigid, if the only continuous of flex of $G(\mathbf{p})$ that satisfies the tensegrity constraints (2) are the restrictions of congruences. One could even call this *local rigidity* as in [20]. There is a good body of work devoted to the detection and understanding of local rigidity. One can see some good surveys in [31, 35, 36].

However, most of the structures that are are made by Snelson and other artists actually enjoy a stronger property. We say a tensegrity $G(\mathbf{p})$ is *globally rigid* in $\mathbb{E}^d$ if for any other configuration $\mathbf{q}$ of the same labeled nodes in $\mathbb{E}^d$, $G(\mathbf{q}) \leq G(\mathbf{p})$ implies that $\mathbf{q}$ is congruent to $\mathbf{p}$. In other words, if the member constraints of (2) are satisfied by $\mathbf{q}$, then there is a rigid congruence of $\mathbb{E}^d$ given by a d-by-d orthogonal matrix A and a vector $\mathbf{b} \in \mathbb{E}^d$ such that for $i = 1, \dots, n$, $\mathbf{q}_i = A\mathbf{p}_i + \mathbf{b}$. Indeed, even more strongly, regard $\mathbb{E}^d \subset \mathbb{E}^D$, for $d \leq D$. If, even though $G(\mathbf{p})$ is in E^d, it is true that $G(\mathbf{p})$ is globally rigid in $\mathbb{E}^D$, for all $D \geq d$, then we say $G(\mathbf{p})$ is *universally globally rigid*. For example, both rigid tensegrities in Figure 2, are universally globally rigid. The example in Figure (3a) is rigid in the plane, but not globally rigid in the plane, since it can fold around a diagonal. Figure (3b) is globally rigid in the plane but not universally globally rigid, since it is flexible in three-space. Figure (3c) is universally globally rigid. These are all bar frameworks.

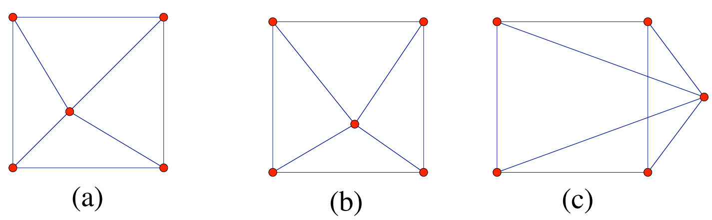


Figure 3: Three examples of planar rigid bar frameworks.

The local and global rigidity of the examples in Figure 3 are fairly easy to determine, but what are some tools to use for more complicated tensegrities? The energy function E_ω described in Section 2 helps. The idea is to look for situations when the configuration $\mathbf{p}$ is a minimum for the functional E_ω. The first step is to determine when $\mathbf{p}$ is a critical point for E_ω. This will happen when all directional derivatives given by $\mathbf{p}' = (\mathbf{p}'_1, \dots, \mathbf{p}'_n)$ starting at $\mathbf{p}$ are 0. So we perform the following calculation starting from (1) for $0 \leq t \leq 1$:

$$E_\omega(\mathbf{p} + t\mathbf{p}') = \sum_{i<j} \omega_{ij}((\mathbf{p}_i - \mathbf{p}_j)^2 + 2t(\mathbf{p}_i - \mathbf{p}_j)(\mathbf{p}'_i - \mathbf{p}'_j) + t^2(\mathbf{p}'_i - \mathbf{p}'_j)^2).$$

Taking derivatives and evaluating at $t = 0$, we get:

$$\frac{d}{dt} E_\omega(\mathbf{p} + t\mathbf{p}')|_{t=0} = 2 \sum_{i<j} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j)(\mathbf{p}'_i - \mathbf{p}'_j). \tag{3}$$

At a critical configuration $\mathbf{p}$, equation (3) must hold for all directions $\mathbf{p}'$, so the following equilibrium vector equation must hold for each node i:

$$\sum_j \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i) = 0. \tag{4}$$

When equation (4) holds for all $i = 1, \ldots, n$, we say ω is an *equilibrium stress* or equivalently a *self stress*, or just a stress for $\mathbf{p}$ when the equilibrium is clear from the context. To get an understanding of how this works, consider the example of a square in the plane as in Figure 4. It is easy to see that the

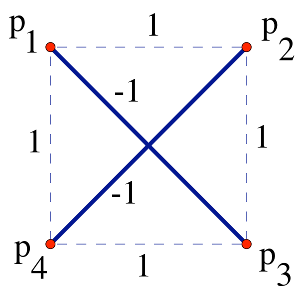


Figure 4: A square tensegrity with its diagonals, where a proper equilibrium stress is indicated.

vector equilibrium equation (4) holds for the three vectors at each node, even though many people tend to put $-\sqrt{2}$ instead of -1 for the strut stresses.

If a configuration $\mathbf{p}$ were the unique minimum, up to rigid congruences, for E_ω, we would have a global rigidity result immediately, but unfortunately that is almost never the case. We must deal with affine transformations.

4 Affine transformations

An *affine transformation* or *affine map* of $\mathbb{E}^d$ is determined by a d-by-d matrix A and a vector $\mathbf{b} \in \mathbb{E}^d$. If $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ is any configuration in $\mathbb{E}^d$, an affine image is given by $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n)$, where $\mathbf{q}_i = A\mathbf{p}_i + \mathbf{b}$.

If the configuration $\mathbf{p}$ is in equilibrium with respect to the stress ω, then so is any affine transformation $\mathbf{q}$ of $\mathbf{p}$, as is seen by the following calculation:

$$\sum_j \omega_{ij}(\mathbf{q}_j - \mathbf{q}_i) = \sum_j \omega_{ij}(A\mathbf{p}_j + \mathbf{b} - A\mathbf{p}_i - \mathbf{b}) = A\sum_j \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i) = 0.$$

So our stress-energy functional E_ω can't "see" affine transformations, at least at critical points. Of course we know that when something is globally rigid, it cannot exclude rigid congruences, but the group of affine transformations are more than we would like. Notice that even projections, which are singular affine transformations, also preserve equilibrium configurations. Indeed, the equilibrium formula (4) is true if and only if it is true for each coordinate, which is the same as being true for orthogonal projections onto each coordinate axis.

This brings us to the question, for a tensegrity $G(\mathbf{p})$ in $\mathbb{E}^d$, when is there an affine transformation that preserves the member constraints (2)? It is clear that the matrix A is the only relevant part. For us, it will turn out that we also only need to consider when the members are bars. If $\{i, j\}$ is a bar of G, then the matrix A determines a transformation that preserves that bar length if and only if the following holds:

$$\begin{aligned}
(\mathbf{p}_i - \mathbf{p}_j)^2 &= (\mathbf{q}_i - \mathbf{q}_j)^2 \\
&= (A\mathbf{p}_i - A\mathbf{p}_j)^2 \\
&= [A(\mathbf{p}_i - \mathbf{p}_j)]^T A(\mathbf{p}_i - \mathbf{p}_j) \\
&= (\mathbf{p}_i - \mathbf{p}_j)^T A^T A(\mathbf{p}_i - \mathbf{p}_j),
\end{aligned}$$

or equivalently,

$$(\mathbf{p}_i - \mathbf{p}_j)^T(A^T A - I^d)(\mathbf{p}_i - \mathbf{p}_j) = 0 \qquad (5)$$

where $()^T$ is the transpose operation, I^d is the d-by-d identity matrix, and vectors are regarded as column vectors in this calculation. If Equation (5) holds for all bars in $G(\mathbf{p})$, we say it has a *bar preserving affine image*, which is non-trival if A is not orthogonal. Similarly, $G(\mathbf{p})$ has a *non-trivial affine flex* if there is a continuous family of d-by-d matrices A_t, where $A_0 = I^d$, for t in some interval containing 0 such that each A_t satisfies Equation (5) for t in the interval.

This suggests the following definition. If $\mathbf{v} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a collection of vectors in $\mathbb{E}^d$, we say that they lie on a *quadric at infinity* if there is a non-zero symmetric d-by-d matrix Q such that for all $\mathbf{v}_i \in \mathbf{v}$

$$\mathbf{v}_i^T Q \mathbf{v}_i = 0. \qquad (6)$$

The reason for this terminology is that real projective space $\mathbb{RP}^{d-1}$ can be regarded as the set of lines through the origin in $\mathbb{E}^d$, and equation (6) is the definition of a quadric in $\mathbb{RP}^{d-1}$.

Notice that since the definition of an orthogonal matrix A is that $A^TA - I^d = \mathbf{0}$, the affine transformation defines a quadric at infinity if and only if the affine transformation is not a congruence.

Call the *bar directions of a bar tensegrity* the set $\{\mathbf{p}_i - \mathbf{p}_j\}$, for $\{i,j\}$ a bar of G. With this terminology, Equation (6) says that if the member directions of a bar tensegrity under an affine transformation A satisfy (5), they lie on a quadric at infinity. Conversely suppose that the member directions of a bar tensegrity $G(\mathbf{p})$ lie on a quadric at infinity in $\mathbb{E}^d$ given by a non-zero symmetric matrix Q. By the spectral theorem for symmetric matrices, we know that there is an orthogonal d-by-d matrix $X = (X^T)^{-1}$ such that:

$$X^TQX = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_d \end{pmatrix}.$$

Let λ_- be the smallest λ_i, and let λ_+ be the largest λ_i. Note $\infty \leq 1/\lambda_- < 1/\lambda_+ \leq \infty$, λ_- is non-positive, and λ_+ is non-negative when Q defines a non-empty quadric and when $1/\lambda_- \leq t \leq 1/\lambda_+$, $1 - t\lambda_i \geq 0$ for all $i = 1, \ldots, d$. Working Equation (5) backwards for $1/\lambda_- \leq t \leq 1/\lambda_+$ we define:

$$A_t = X^T \begin{pmatrix} \sqrt{1-t\lambda_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{1-t\lambda_2} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{1-t\lambda_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sqrt{1-t\lambda_d} \end{pmatrix} X. \quad (7)$$

Substituting A_t from Equation (7) into Equation (5), we see that it provides a non-trivial affine flex of $G(\mathbf{p})$. If the configuration is contained in a lower dimensional hyperplane, we should really restrict to that hyperplane since there are non-orthogonal affine transformations that are rigid when restricted to the configuration itself. We have shown the following:

Proposition 1. *If $G(\mathbf{p})$ is a bar framework in $\mathbb{E}^d$, such that the nodes do not lie in a $(d-1)$-dimensional hyperplane, then it has a non-trivial bar preserving* *affine image if and only if it has a non-trivial bar preserving affine flex if and only if the bar directions lie on a quadric at infinity.*

It is useful to consider when bar tensegrities have the bar directions that lie on a quadric at infinity. In $\mathbb{E}^2$, the quadric at infinity consists of two distinct directions. So a parallelogram or a grid of parallelograms have a non-trivial affine flex. In $\mathbb{E}^3$ it is more interesting. The quadric at infinity is a conic in $\mathbb{RP}^2$, the projective plane, and such a conic is determined by 5 points. An interesting example is the bar tensegrity in Figure 5. The surface

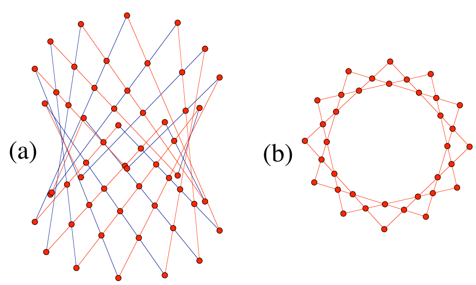


Figure 5: Figure (a) is the ruled hyperboloid given by $x^2 + y^2 - z^2 = 1$. Figure (b) is the flattened version after an affine flex.

is obtained by taking the line $(x, 1, x)$ and rotating it about the z-axis. This creates a ruling of the surface by disjoint lines. Similarly $(x, 1, -x)$ creates another ruling. Each line in one ruling intersects each line in the other ruling or they are parallel. A bar tensegrity is obtained by placing nodes where a line on one ruling intersects a line on the other ruling, and bars such that they join every pair of nodes that lie on same line on either ruling.

Consider the diagonal matrix Q with diagonal entries $\lambda_1 = \lambda_2 = 1, \lambda_3 = -1$. When one node of each bar is translated to a single point, they all lie on a circle at infinity given by Q. The flex given by Formula (7) flexes the configuration until the nodes lie on a line when $t = 1/\lambda_+ = 1$ because two of the eigenvalues for Q vanish for that value of t, and in the other direction, when $t = 1/\lambda_- = -1$, the nodes lie in a plane. There is a pleasant description of this motion in [23] at the end of Chapter 1. This structure is easy to build with dowel rods and rubber bands securing the joints where the rulings intersect.

The space of d-by-d symmetric matrices is of dimension $d + (d^2 - d)/2 = d(d+1)/2$. So if the vector directions of a tensegrity are less than $d(d+1)/2$, then it is possible to find a non-zero d-by-d symmetric matrix that satisfies Equation (6), and then flex it into a lower dimensional subspace. This proves the following, due to Barvinok in [2].

Theorem 2. *If $G(\mathbf{p})$ is a bar framework in $\mathbb{E}^D$ with less than $d(d+1)/2$ bars, then it has a realization in $\mathbb{E}^d$ with the same bar lengths.*

This suggests the following definition: If a (bar) graph G is such that any realization $G(\mathbf{p})$, for a configuration $\mathbf{p}$ in some $\mathbb{E}^D$, implies that G have a realization in $\mathbb{E}^d$ with the same bar lengths, then we say G is *d-realizable*. Note that this is a property of the graph G, and in order to qualify for being d-realizable, one has to be able to push a realization in $\mathbb{E}^D$ down to a realization $\mathbb{E}^d$ for ALL realizations in $\mathbb{E}^D$. For example, the 1-realizable graphs are forests, graphs with no cycles. In particular, a triangle is not 1-realizable.

This is inspired from a problem in nuclear magnetic resonance (NMR) spectroscopy. The atoms of a protein are tagged and some of the pairwise distances are known. The problem is to identify a configuration in $\mathbb{E}^3$ that satisfies those distance constraints. Finding such a configuration in $\mathbb{E}^D$, for some large D is computationally feasible, and if G is 3-realizable, one can expect to find another configuration in $\mathbb{E}^3$ that satisfies the distance constraints.

For any graph G, another graph H is called a *minor* of G if it can be obtained by edge contractions or deletions. In particular if a minor of a graph G is not d-realizable, then G itself is not d-realizable. It is easy to see that a graph is 1-realizable if and only if it does not have a triangle as a minor. In other words, the triangle is a complete list of *forbidden minors* for 1-realizability. It is not too hard to show that the graph K_4, the tetrahedron, is also a complete list of forbidden minors for 2-realizability. In [4, 3] M. Belk and I show the following:

Theorem 3. *A complete list of forbidden minors for 3-realizability is the set of two graphs, K_5 and edge graph of the regular octahedron.*

This implies that there is a reasonable algorithm to detect 3-realizablity for an abstract graph and, when the edge lengths are given, to find a realization in $\mathbb{E}^3$. Tensegrity techniques are used in a significant way in [3].

5 The stress matrix

The stress-energy function E_ω defined by (1) of Section 2 is really a quadratic form. It is an easy matter to compute the (symmetric) matrix associated to that quadratic form. Note that the square of the difference of two variables, x_1 and x_2 is

$$(x_1 - x_2)^2 = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

For each $i \neq j$, define an n-by-n matrix $\Omega(i,j)$, where the (i,i) and (j,j) entries are 1, the (i,j) and (j,i) entries are -1, while are the other entries are 0. Then

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \sum_{i<j} \omega_{ij}\Omega(i,j) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i<j} \omega_{ij}(x_i - x_j)^2,$$

for any stress $(\cdots, \omega_{ij}, \cdots)$. So for any stress ω, where $\omega_{ij} = \omega_{ji}$ for all $1 \leq i \leq j \leq n$, define the associated n-by-n *stress matrix* $\Omega = \sum_{i<j} \omega_{ij}\Omega(i,j)$ such that the (i,j) entry is $-\omega_{ij}$ for $i \neq j$, and the diagonal entries are such that the row and column sums are 0. Recall that any stress ω_{ij} not designated in the vector form $\omega = (\cdots, \omega_{ij}, \cdots)$ is assumed to be 0.

With this terminology regard a configuration $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$ in $\mathbb{E}^d$ as a column vector. Then

$$\begin{aligned} E_\omega(\mathbf{p}) &= \sum_{i<j} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j)^2 \\ &= \sum_{i<j} \omega_{ij}(x_i - x_j)^2 + \sum_{i<j} \omega_{ij}(y_i - y_j)^2 + \dots \\ &= \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \Omega \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix} \Omega \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \dots, \end{aligned}$$

where each $\mathbf{p}_i = (x_i, y_i, \dots)$, for $i = 1, \dots, n$. So we see that E_ω is essentially given by the matrix Ω repeated d times. The tensor product of matrices (or sometimes the Kronecker product) gives the matrix of E_ω as $\Omega \otimes I^d$, and

$$E_\omega(\mathbf{p}) = (\mathbf{p})^T \Omega \otimes I^d \mathbf{p}.$$

It is also convenient to rewrite the equilibrium condition (4) in terms of matrices. Define the *configuration matrix* for the configuration $\mathbf{p}$ as

$$P = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

a $(d+1)$-by-n matrix, and the equilibrium condition (4) is equivalent to

$$P\Omega = \mathbf{0}.$$

Each coordinate of P as a row vector multiplied on the right by Ω represents the equilibrium condition in that coordinate. The last row of ones of P represent the condition that the column sums (and therefore the row sums) of Ω are $\mathbf{0}$. It is also easy to see that the linear rank of P is the same as the dimension of the affine span of $\mathbf{p}_1, \ldots, \mathbf{p}_n$ in $\mathbb{E}^d$.

Suppose that we add rows to P until all the rows span the co-kernel of Ω. The corresponding configuration $\mathbf{p}$ will be called a *universal configuration* for ω (or equivalently Ω).

Proposition 4. *If* $\mathbf{p}$ *is a universal configuration for* ω*, any other configuration* $\mathbf{q}$ *which is in equilibrium with respect to* ω *is an affine image of* $\mathbf{p}$*.*

Proof. Let Q be the configuration matrix for $\mathbf{q}$. Since the rows of P are a basis for the co-kernel of Ω, and the rows of Q are, by definition, in the co-kernel of Ω, there is a $(d+1)$-by-$(d+1)$ matrix A such that $AP = Q$. Since P and Q share the last row of ones, we know that A takes the form

$$A = \begin{pmatrix} A_0 & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix},$$

where A_0 is a d-by-d matrix, $\mathbf{b}$ is a 1-by-d matrix (a vector in $\mathbb{E}^d$), and the last row is all 0's except for the 1 in the lower right hand entry. Then we see that for each $i = 1, \ldots, n$, $\mathbf{q}_i = A_0\mathbf{p}_i + \mathbf{b}$, as desired. □

The stress matrix plays a central role in what follows. Note that when the configuration $\mathbf{p}$ is universal, with affine span all of $\mathbb{E}^d$, for the stress ω, the dimension of the co-kernel (which is the dimension of the kernel) of Ω is d, and the rank of Ω is $n-d-1$. But even when the configuration $\mathbf{p}$ is not universal for ω, it is the projection of a universal configuration, and so the rank $\Omega \le n-d-1$.

6 The fundamental theorem

We come to one of the basic tools for showing specific tensegrities are globally rigid and more. If ω is a proper equilibrium stress for the tensegrity $G(\mathbf{p})$, $\mathbf{p}_i - \mathbf{p}_j$, where $\omega_{ij} \neq 0$ is called a *stressed direction* and the member $\{i, j\}$ is called a *stressed member*. Note that if $G(\mathbf{q}) \leq G(\mathbf{p})$, $\omega_{ij} \neq 0$, and $|\mathbf{p}_i - \mathbf{p}_j| \neq |\mathbf{q}_i - \mathbf{q}_j|$, then $E_\omega(\mathbf{q}) < E_\omega(\mathbf{p})$. So if $\mathbf{p}$ is a configuration for the minimum of E_ω, the stressed members are effectively bars. This allows Proposition 1 to be applied.

Theorem 5. *Let $G(\mathbf{p})$ be a tensegrity, where the affine span of $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ is all of $\mathbb{E}^d$, with a proper equilibrium stress ω and stress matrix Ω. Suppose further*

1.) Ω is positive semi-definite.

2.) The configuration $\mathbf{p}$ is universal with respect to the stress ω. (In other words, the rank of Ω is $n - d - 1$.)

3.) The stressed directions of $G(\mathbf{p})$ do not lie on a quadric at infinity.

Then $G(\mathbf{p})$ is universally globally rigid.

Proof. Suppose that $\mathbf{q}$ is configuration such that $G(\mathbf{q}) \leq G(\mathbf{p})$. Then $E_\omega(\mathbf{q}) \leq E_\omega(\mathbf{p})$. By Condition 1.), $E_\omega(\mathbf{q}) = E_\omega(\mathbf{p}) = 0$, and ω is an equilibrium stress for the configuration $\mathbf{q}$ as well as $\mathbf{p}$. By Condition 2.) and Proposition 4, $\mathbf{q}$ is an affine image of $\mathbf{p}$. By Condition 3.) and Proposition 1, $\mathbf{q}$ is congruent to $\mathbf{p}$. □

Notice that in view of Proposition 1, Condition 3.) can be replaced by the condition that $G(\mathbf{p})$ is has no affine flexes in $\mathbb{E}^d$. For example, if it is rigid in $\mathbb{E}^d$, that would be enough.

With this in mind, we say that a tensegrity is *super stable* if it has a proper equilibrium stress ω such that Conditions 1.), 2.) and 3.) hold. If just Conditions 1.) and 3.) hold and ω is strict (all members stressed), then we say $G(\mathbf{p})$ is *unyielding*. An unyielding tensegrity, essentially, has all its members replaced by bars.

7 Examples

7.1 The square tensegrity

The stress matrix for the square of Figure 4 is

$$\Omega = \begin{pmatrix} +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \\ +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \end{pmatrix}.$$

So Ω has rank $1 = 4 - 2 - 1 = n - d - 1$, and since its trace is 4, its single eigenvalue is 4, and it is positive semi-definite. This makes it unyielding, and since the underlying graph is the complete graph, it is universally globally rigid. It is also super stable. There are several ways to generalize this example.

7.2 Polygon tensegrities

In [9] I showed that a tensegrity, obtained from a planar convex polygon by putting a node at each vertex, a cable along each edge, and struts connecting other nodes such that the resulting tensegrity has some proper equilibrium stress, is always super stable. Figure 6 shows some examples. These results answered some questions of Grünbaum in his notes [21].

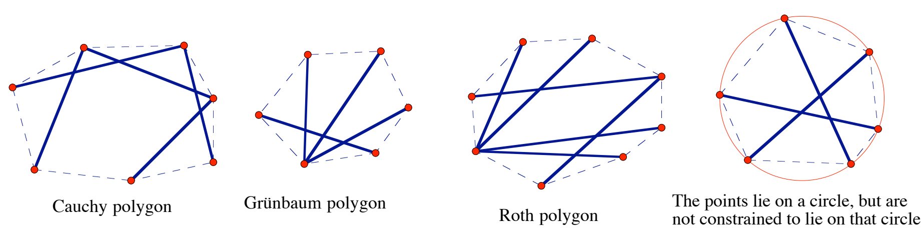


Figure 6:

7.3 Radon tensegrities

Radon's Theorem says that if $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_{d+2})$ are $d+2$ points in $\mathbb{E}^d$, no $d+1$ in a hyperplane, then they can be separated into two simplices σ^i and σ^{d-i} of dimension i and $d-i$ such that their intersection is a common point, which is a relative interior point of each simplex. They can also be used to define a super stable tensegrity as well. Write $\sum_{k=1}^{d+2}\lambda_k\mathbf{p}_k = \mathbf{0}$, where $\sum_{k=1}^{d+2}\lambda_k = 0$, while $\lambda_k > 0$ for $k = 1,\ldots,i+1$, and $\lambda_k < 0$ for $k = i+2,\ldots,d+2$. Then the following rank one matrix is the stress matrix

$$\Omega = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{d+2} \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{d+2} \end{pmatrix},$$

since for the configuration matrix P,

$$P \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{d+2} \end{pmatrix} = \mathbf{0}.$$

So the stress $\omega_{ij} = -\lambda_i\lambda_j$. The edges of σ^i and σ^{d-i} are struts, while all the other members are cables. Since the rank is $d+2-d-1=1$, and Ω is positive semi-definite, the tensegrity is super stable. Figure 7 shows the two examples in the plane and in three-space.

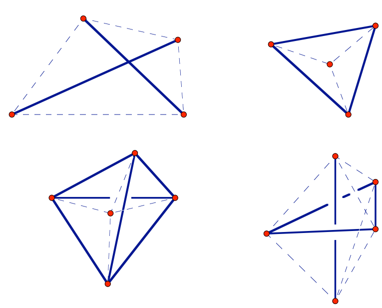


Figure 7:

This tensegrity has been described in [7].

7.4 Centrally symmetric polyhedra

In [29] L. Lovasz showed effectively that if one places nodes at the vertices of a centrally symmetric convex polytope, cables along its edges, and struts between its antipodal points, the resulting tensegrity has a strict proper equilibrium stress, and any such stress will have a stress matrix such that Conditions 1.) and 2.) hold, while condition 3.) is easy to check. Thus such a tensegrity is super stable and universally globally rigid. This is explained in [6] and answers a question of K. Bezdek. Figure 8 shows such an example for the cube, which is easy to check independently.

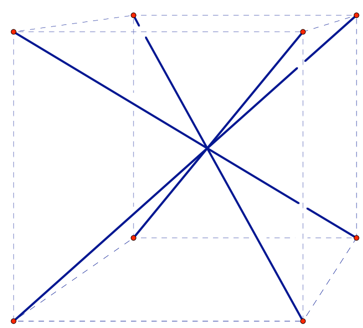


Figure 8: A cube with cables along its edges and struts connecting antipodal nodes, which is super stable.

7.5 Prismatic tensegrities

Consider a tensegrity in $\mathbb{E}^3$ formed by two regular polygons $(\mathbf{p}_1, \dots, \mathbf{p}_n)$ and $(\mathbf{p}_{n+1}, \dots, \mathbf{p}_{2n})$ in distinct parallel planes, each symmetric about the same axis. Cables are placed along the edges of each polygon. Each node of each polygon in connected by a cable to a corresponding node in the other polygon, maintaining the rotational symmetry. Similarly each node of each polygon in connected to a corresponding node in the other polygon by a strut, maintaining the rotational symmetry. The ends of the cable and strut are k steps apart where $1 \leq k \leq n-1$. This describes a *prismatic tensegrity* $P(k, n)$. In [16] it is shown that each $P(n, k)$ is super stable when the angle of the twist from a node in the top polygon to the projection of the node at the other end of the strut is $\pi(1/2 + i/n)$. Figure 9 shows $P(6, 1)$. The Snelson tensegrity in the introduction is $P(3, 1)$.

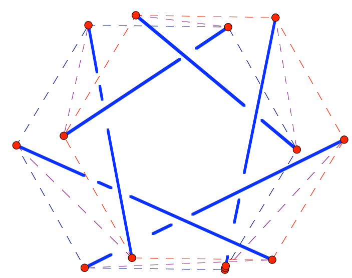


Figure 9: The prismic tensegrity $P(6,1)$.

7.6 Highly symmetric tensegrities

Many of the tensegrities created by artists such as Snelson have the super stable property discussed here. They need the stress for their stability. Their tensional integrity is part of their stability. Symmetry seems to a natural part of art, so I thought it would be interesting to see what symmetric tensegrities were super stable. It turns out that the symmetry simplifies the calculation of the rank and definiteness of the stress matrix. In addition the theory of the representations of finite groups is a natural tool that can be used to decompose the stress matrix. With Allen Back and later Robert Terrell, we created a website, where one can view and rotate the pictures of these tensegrities. This is available at [40]. See also [11] for an explanation of the group theory and rigidity theory.

The tensegrity graph G is chosen so that there is an underlying finite group Γ acting on the tensegrity such that the action of Γ takes cables to cables and struts to struts, and the following conditions hold:

i.) The group Γ acts transitively and freely on the nodes. In other words, for each $\mathbf{p}_i, \mathbf{p}_j$ nodes, there is a unique element $g \in \Gamma$ such that $g\mathbf{p}_i = \mathbf{p}_j$.

ii.) There is one transitivity class of struts. In other words, if $\{\mathbf{p}_i, \mathbf{p}_j\}$ and $\{\mathbf{p}_k, \mathbf{p}_l\}$ are struts, then there is $g \in \Gamma$ such that $\{g\mathbf{p}_i, g\mathbf{p}_j\} = \{\mathbf{p}_k, \mathbf{p}_l\}$ as sets.

iii.) There are exactly two transitivity classes of cables. In other words, all the cables are partitioned into two sets, where Γ permutes the the elements of each set transitively, but no group element takes a cable from one partition to the other.

The user must choose the abstract group, the group elements that correspond to the cables, the group element that corresponds to the struts, and the ratio of the stresses on the two classes of cables. Then the tensegrity is rendered. Figure 10 shows a typical picture from the catalog.

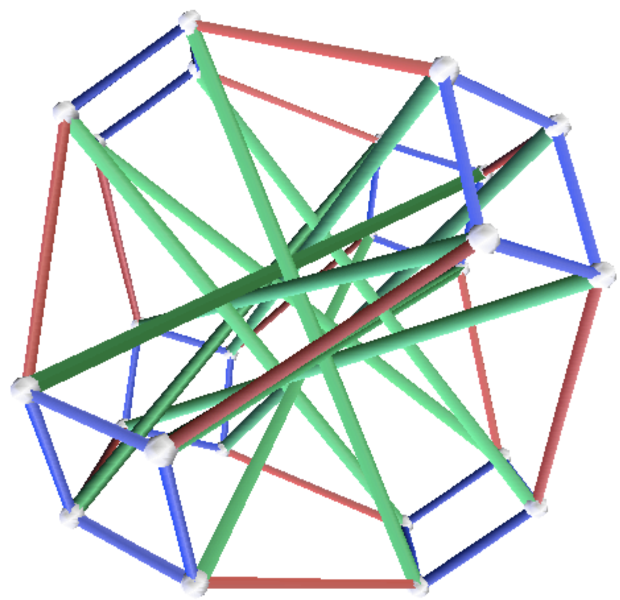


Figure 10: A super stable tensegrity from the catalog [40]. In the catalog, the struts are colored green, one cable transitivity class is colored red, and the other blue. In this example, the cables lie on the convex hull of the nodes, and struts are inside.

7.7 Compound tensegrities

The sum of positive semi-definite matrices is positive semi-definite. So we can glue two super stable tensegrities along some common nodes, and maintain Condition 1.). Condition 3.) is no problem. But the rank Condition 2.) may be violated. But even if Condition 2.) does not hold, each of the individual tensegrities will remain globally rigid, even if some of the stresses vanish on overlapping members.

One example of this process is the delta-Y transformation. If one super stable tensegrity has a triangle of cables in it, one can add a tensegrity of the form in the upper right of Figure 7 so the stresses on the overlap of the three struts exactly cancel with the three cable stresses in the other tensegrity. So the three triangle cables are replaced the three other cables joined to a new node inside the triangle. In this case the resulting tensegrity is still super stable since the radon tensegrity is planar and using Condition 3.). Figure 11 shows how this might work for the top triangle of the Snelson tensegrity of Figure 2. Taken from [39] Figure 12 shows this replacement on both triangles.

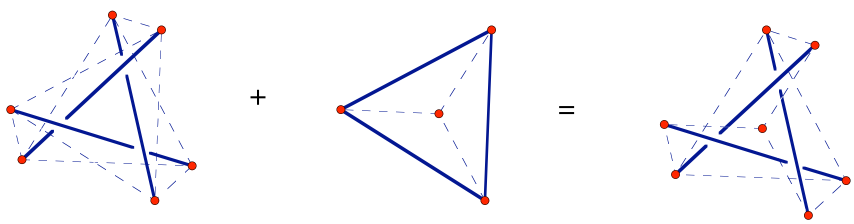


Figure 11:

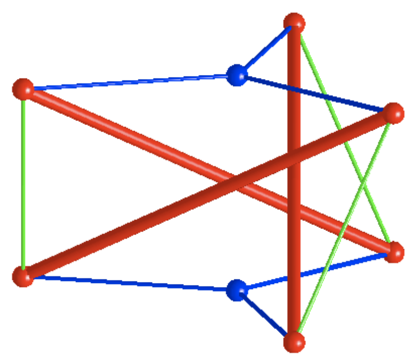


Figure 12:

If the replacement as in Figure 12 is done for a polygon of with four or more vertices, the resulting tensegrity may not be super stable or even rigid, but in [39] it is shown that if the polygons have an odd number of vertices and the struts are placed as far away from the vertical cables as possible, then the resulting tensegrity is super stable. In other words, if the star construction is done on $P(2k+1,k)$ as in Figure 13 the resulting tensegrity is super stable.

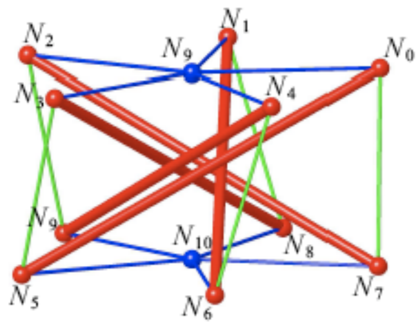


Figure 13:

It is also possible to put two (or more) super stable together on a common polygon to create a tensegrity with a stress matrix that satisfies Condition 1.) while the universal configuration is 4-dimensional instead of 3-dimensional. But each of the original pieces are universally globally rigid. The 4-dimensional realization has an affine flex around the 2-dimensional polygon used to glue the two pieces together. So the tensegrity has two non-congruent configurations in $\mathbb{E}^3$ as one piece rotates about the other in $\mathbb{E}^4$. Meanwhile struts and cable stresses can be arranged to cancel, and thus those members are not needed in the compound tensegrity. Figure 14 shows this with two Snelson tensegrities combined along a planar hexagonal tensegrity. This is similar to the situation in [26] by T. Jordan and Z. Szabadka, where some pairs of nodes do not change their distances for other non-congruent realizations.

This is something like the start of the Snelson tower of Figure 1, but hexagonal polygon in the middle is planar, which seems a bit surprising. This tensegrity is unyielding and rigid, but not super stable. But possibly to create more stability Snelson includes more cables from one unit to the other, and this destroys the planarity of the hexagon.

There are many different ways to combine super stable units, possibly erasing some of the members in the basic units to get similar rigid tensegrities.

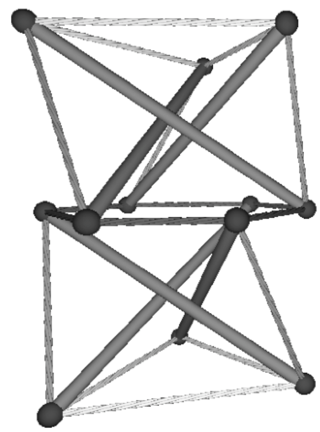


Figure 14:

7.8 Pure and flexible examples

Recall that a pure tensegrity is one that has only cables and struts and the struts are all disjoint. We have seen several examples in $\mathbb{E}^3$ of pure tensegrities, the simplest being Snelson's original as in Figure 11 on the left. But what about the plane? One might be tempted to think that the tensegrity of Figure 15 is rigid, but it isn't. Indeed, there are no pure rigid

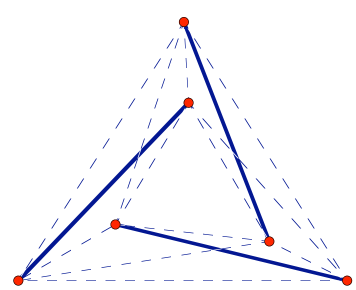


Figure 15:

tensegrities in the plane. This follows from the main theorem in [14], the proof of the carpenter's rule property. This theorem says that any chain of non-overlaping edges in the plane can be continuously expanded (flexed) until it is straight. This result also allows for disjoint edges, and at least, for a short time, the expansion can be run backwards to be a contraction, keeping the struts at a fixed length. See [13] for a discussion of expansive flexes from a tensegrity point of view.

8 Generic global rigidity

The configurations in previous sections must be constructed carefully. What about a bar framework where the configuration is more general? It turns out that the problem of determining when a bar framework is globally rigid is equivalent to a long list of problems known to be hard. See [33] for example. The problem of whether a cyclic chain of edges in the line has another realization with the same bar lengths, is equivalent to the uniqueness of a solution of the knapsack problem. This is one of the many problems on the list of NP complete problems.

One way to avoid this difficulty, is to assume that the configuration's coordinates are *generic*. This means that the coordinates of $\mathbf{p}$ in $\mathbb{E}^d$ are *algebraically independent over the rational numbers*, which means that there is no non-zero polynomial with rational coordinates satisfied by the coordinates of $\mathbf{p}$. This implies, among other things, that no $d+2$ nodes lie in a hyperplane, for example, and a lot more. In [12] I proved the following:

Theorem 6. *If $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$ in $\mathbb{E}^d$ is generic and $G(\mathbf{p})$ is a rigid bar tensegrity in $\mathbb{E}^d$ with a non-zero stress matrix Ω of rank $n-d-1$, then $G(\mathbf{p})$ is globally rigid in $\mathbb{E}^d$.*

Notice that the hypothesis includes Conditions 2 and 3 of Theorem 5. The idea of the proof is to show that since the configuration $\mathbf{p}$ is generic, if $G(\mathbf{q})$ has the same bar lengths as $G(\mathbf{p})$, then they should have the same stresses. Then Proposition 1 applies.

Then recently in [20] the converse of Theorem 6 was shown as follows:

Theorem 7. *If $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$ in $\mathbb{E}^d$ is generic and $G(\mathbf{p})$ is a globally rigid bar tensegrity in $\mathbb{E}^d$, then either $G(\mathbf{p})$ is a bar simplex or there is stress matrix Ω for $G(\mathbf{p})$ with rank $n-d-1$.*

The idea here, very roughly, is to show that a map from an appropriate quotient of an appropriate portion of the space of all configurations has even topological degree when mapped into the space of edge lengths.

As pointed out in [20], using these results it is possible to find a polynomial time numerical (probablistic) algorithm that calculates whether a given graph is generically globally rigid in $\mathbb{E}^d$, and that the property of being globally rigid is a generic property. In other words, if $G(\mathbf{p})$ is globally rigid in $\mathbb{E}^d$ at one generic configuration $\mathbf{p}$, it is globally rigid at all generic configurations.

Interestingly, it is also shown in [20] that if $\mathbf{p}$ is generic in $\mathbb{E}^d$, and $G(\mathbf{q})$ has the same bar lengths in $G(\mathbf{p})$ in $\mathbb{E}^d$, then $G(\mathbf{p})$ can be flexed to $G(\mathbf{q})$ in $\mathbb{E}^{d+1}$, similar to the tensegrity mentioned in Subsection 7.7 of compounded Snelson tensegrities.

A bar graph G is defined to be *generically redundantly rigid* in $\mathbb{E}^d$ if $G(\mathbf{p})$ is rigid at a generic configuration $\mathbf{p}$, and it remains rigid after the removal of any bar. A graph is vertex k-connected if it takes the removal of at least k vertices to disconnect the rest of the vertices of G. The following theorem of Hendrickson [22], provides two necessary conditions for generic global rigidity.

Theorem 8. *If* $\mathbf{p}$ *is a generic configuration in* $\mathbb{E}^d$*, and the bar tensegrity* $G(\mathbf{p})$ *is globally rigid in* $\mathbb{E}^d$*, then*

a.) G *is vertex* $(d+1)$*-connected, and*

b.) $G(\mathbf{p})$ *is redundantly rigid in* $\mathbb{E}^d$*.*

Condition a.) on vertex connectivity is clear since otherwise it is possible to reflect one component of G about the hyperplane determined by some d or fewer vertices. Condition b.) on redundant rigidity is natural since if, after a bar $\{\mathbf{p}_i, \mathbf{p}_j\}$ is removed, $G(\mathbf{p})$ is flexible, one watches as the distance between $\mathbf{p}_i$ and $\mathbf{p}_j$ changes during the flex, and waits until the distance comes back to it original length. If $\mathbf{p}$ is generic to start with, the new configuration will be not congruent to the original configuration.

Hendrickson conjectured that Conditions a.) and b.) were also sufficient for generic global rigidity, but it turns out in [10] that the complete bipartite graph $K_{5,5}$ in $\mathbb{E}^3$ is a counterexample. This is easy to see as follows.

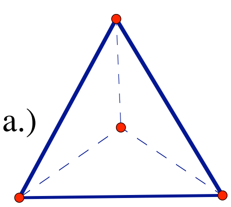
Similar to the analysis in Subsection 7.3 for each of the nodes for the two partitions of $K_{5,5}$ consider the affine linear dependency $\sum_{i=1}^{5} \lambda_i \mathbf{p}_i = \mathbf{0}, \sum_{i=1}^{5} \lambda_i = 0$ and $\sum_{i=6}^{10} \lambda_i \mathbf{p}_i = \mathbf{0}, \sum_{i=6}^{10} \lambda_i = 0$, where $(\mathbf{p}_1, \dots, \mathbf{p}_5)$ and $(\mathbf{p}_6, \dots, \mathbf{p}_{10})$ are the two partitions of $K_{5,5}$. According to [8], when the configuration $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_{10})$ is generic in $\mathbb{E}^3$, then, up to a scaling factor, the stress matrix for $K_{5,5}(\mathbf{p})$ is

$$\Omega = \begin{pmatrix} \mathbf{0} & \begin{pmatrix} \lambda_6 \\ \vdots \\ \lambda_{10} \end{pmatrix} \begin{pmatrix} \lambda_1 & \cdots & \lambda_5 \end{pmatrix} \\ \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_5 \end{pmatrix} \begin{pmatrix} \lambda_6 & \cdots & \lambda_{10} \end{pmatrix} & \mathbf{0} \end{pmatrix}.$$

But the rank of Ω is $2 < 10 - 3 - 1 = 6$, while rank 6 is needed for generic global rigidity in this case by Theorem 7.

In $\mathbb{E}^3$, $K_{5,5}$ is the only counterexample to Hendrickson's conjecture that I know of. On the other hand in [18] it is shown that a graph G is generically globally rigid $\mathbb{E}^d$ if an only if the cone over G is generically globally rigid in $\mathbb{E}^{d+1}$. This gives more examples in dimensions greater than 3, and there are some other bipartite graphs as well in higher dimensions by an argument similar to the one here.

Meanwhile, the situation in the plane is better. Suppose G is a graph and $\{i, j\}$ is an edge of G, determined by nodes i and j. Remove this edge, add another node k and join k to i, j, and $d-1$ distinct other nodes not i or j. This is called a *Henneberg operation* or sometimes *edge splitting*. Figure 16 shows this operation on a tensegrity (although we are concerned here with bar frameworks).

a.)


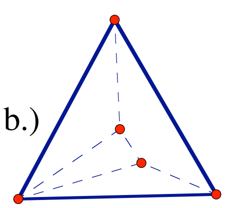
b.)


Figure 16: Graph a.) is split along the lower right inside edge, to get the graph b.).

It is not hard to show that edge splitting, as in Figure 16 preserves generic global rigidity in $\mathbb{E}^d$. When the added node lies in the relative interior of the line segment of the bar that is being split, there is a natural stress for the new bar tensegrity, and the subdivided tensegrity is also universal with respect to the new stress. If the original configuration is generically rigid, a small perturbation of the new configuration to a generic one will not change the rank of the stress matrix. Thus generic global rigidity is preserved under edge splitting. In [5, 24], A. Berg and T. Jordán and later B. Jackson and T. Jordán solve a conjecture of mine with the following:

Theorem 9. *If a graph G is vertex 3-connected (Condition a.) for $d = 2$) and is generically redundantly rigid in the plane (Condtion b.) for $d = 2$), then G can be obtained from the graph K_4, by a sequence of edge splits (as in Figure 16) and insertions of additional bars.*

Thus Hendrickson's conjecture, that Condition a.) and Condtion b.) are sufficient as well as necessary for generic global rigidity in the plane, is sufficient as well as necessary. This also gives an efficient non-probablistic polynomial-time algorithm for determining generic global rigidity in the plane. See also [25, 28, 30, 27] for other results and applications.

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