Prestress stability

Lecture VI

Session on Granular Matter
Institut Henri Poincaré

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Potential functions

How is the stability of a structure determined when it is statically indeterminant or even not statically rigid?

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Think of each member as a spring at rest (a bar), in tension (a cable), or in compression (a strut). In the spirit of Hooke's Law define an energy function as follows:

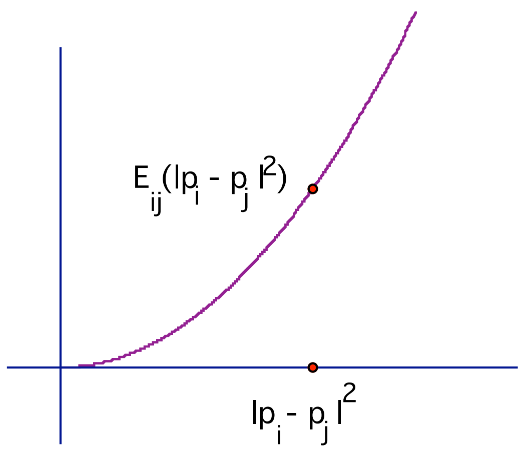
Member potentials

For each member {i,j} define (or determine) $E_{ij}(x)$ its potential at length $x^{1/2}$. So the total energy for any configuration $\mathbf{q}=(q_1, q_2, \ldots q_n)$ in $\mathbf{E}^d$ is

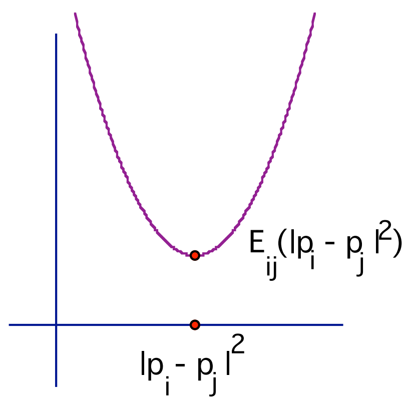
$$E(\mathbf{q})=\sum_{i<j} E_{ij}(|q_i-q_j|^2).$$

The E_{ij} functions are chosen (or computed) with respect to a fixed configuration **p** as follows.

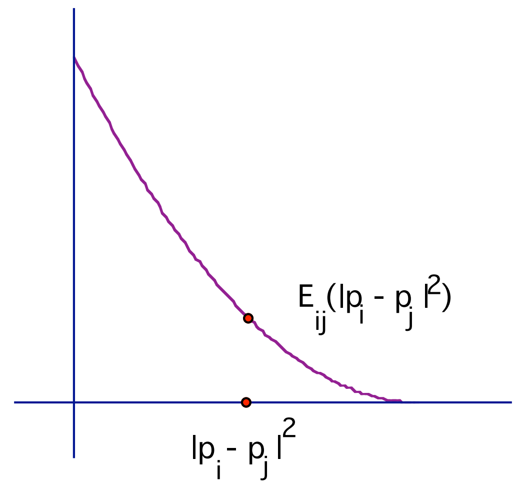
The individual energy functions



A cable: Monotone increasing, concave up



A bar: A local minimum, concave up



A strut: Monotone decreasing, concave up

The critical calculation

Define $\square_{ij} = E\square_{ij}(|p_i - p_j|^2)$ the first derivative of E_{ij} at $x = |p_i - p_j|^2$. Then a calculation shows that **p** is a critical point for the energy functional E if and only if $\square = (\ldots, \square_{ij}, \ldots)$ is a proper self stress for the tensegrity G(**p**).

Our goal is to determine when the configuration **p** is a local minimum for E, up to congruences. So we calculate the second derivative.

Stiffness

For each member {i,j}, define

$$c_{ij} = E''_{ij}(|p_i-p_j|^2) > 0,$$

the second derivative of E_{ij} at $|p_i-p_j|^2$ as the *stiffness coefficient* of member {i,j}. We assume that these coefficients are all positive. Define C as the e-by-e diagonal matrix whose ij, ij diagonal entry is c_{ij}, where e is the number of members in the tensegrity.

Hessian

Let $\mathbf{p}' = (p_1', p_2', \ldots p_n')$ be a set of directions, where each p_i' is in $\mathbf{E}^d$. So $E(\mathbf{p} + t\mathbf{p}')$, for $0 \leq t$, has derivative 0 when $t = 0$, for all $\mathbf{p}'$. A calculation shows that the second derivative in this direction $\mathbf{p}'$ is given by the sum of the following two quadratic forms:

$$4 \sum_{i\leq j} c_{ij}[(p_i - p_j)(p_i' - p_j')]^2 + 2 \sum_{i\leq j} \omega_{ij}(p_i' - p_j')^2.$$

The matrix $S = 4\, R(\mathbf{p})^T C R(\mathbf{p})$ is what we call the *stiffness matrix*, where $R(\mathbf{p})$ is our old friend the rigidity matrix. Note that S is always positive semi-definite, and $4\, (\mathbf{p}')^T S\, \mathbf{p}'$ is the term on the left.

Stress matrix

The matrix of the form on the right is 2 □, where each non-zero off-diagonal entry of □ is defined to be $-\Box_{ij}$ corresponding to each coordinate, and such that the row and column sums are 0. The matrix □ is called the *stress matrix* and is very interesting. It can provide a lot of global information about the tensegrity, although it can have negative eigenvalues.

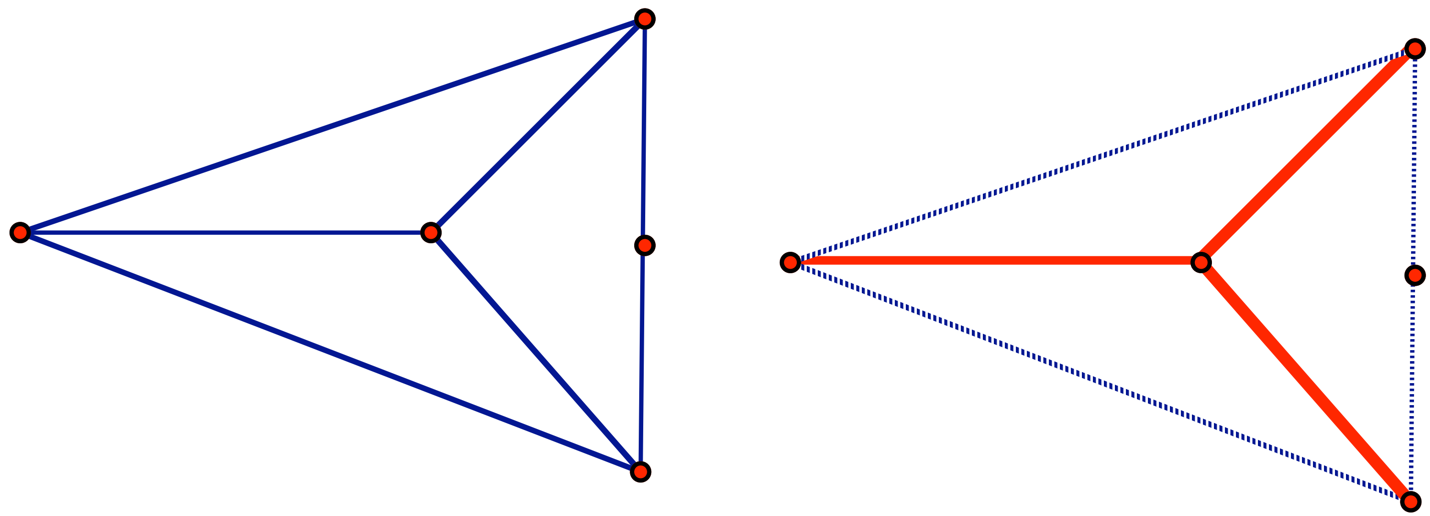
Define the matrix H = 4 S + 2 □. So 2 (**p**□)[□] □ **p**□ is the term on the right for the second derivative above.

Prestress stability

Corresponding to given stress coefficients □, and stiffness coefficients, we say that G(**p**) is *prestress stable* if H is positive semi-definite with only the trivial infinitesimal flexes in its kernel. The following is in Connelly-Whiteley 1995.

Theorem: If a tensegrity G(**p**) is prestress stable in $\mathbf{E}^d$, then it is rigid in $\mathbf{E}^d$.

Examples of prestress stable tensegrities



The pattern of tensions and compressions is on the right. Neither of these structures is infinitesimally rigid.

Displacements and forces

Given a prestress stable tensegrity structure G(**p**) and an equilibrium force **F** applied to the configuration **p**, how does **p** deform to accommodate **F**, and how is the force resolved?

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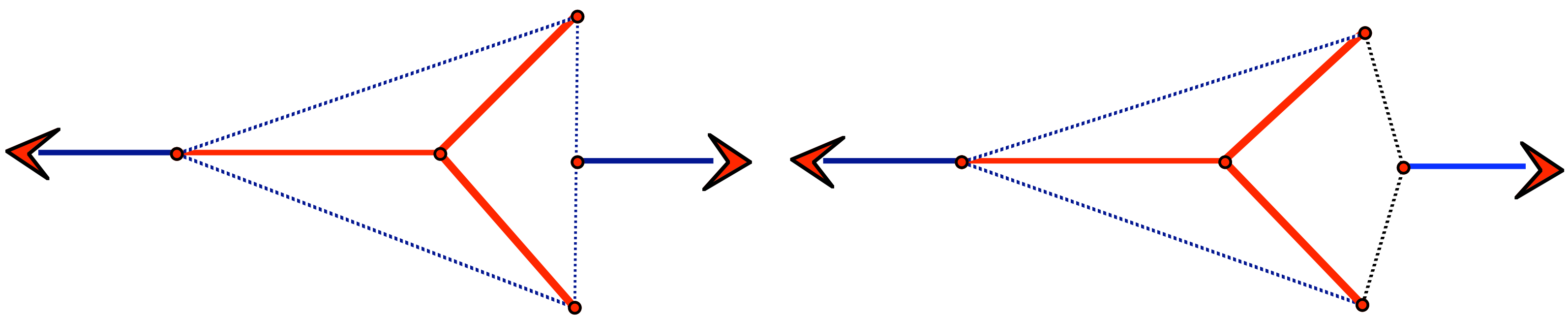
Given a prestress stable tensegrity framework G(**p**) and an equilibrium force **F** applied to the configuration **p**, how does **p** deform to accommodate **F**, and how is the force resolved?

As a first approximation to the displacement □**p**, solve **F** + 2H□**p** = 0, which is possible since H has maximal rank, and 2H is the gradient of the approximation to the energy form. Then the resolving stress can itself be determined from the edge lengths and the energy formulas.

Example of a displacement



When the equilibrium force above is applied to this tensegrity it cannot resolve the force as is, but. . .

it deforms into a configuration that does resolve the force.

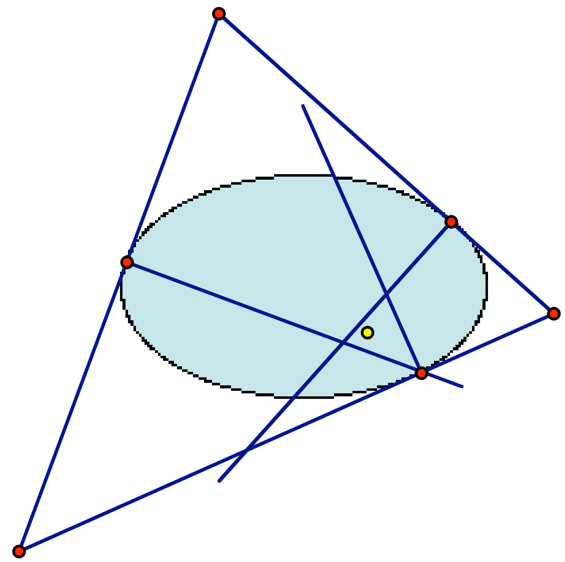
An application to packings

What sort of model of rigidity is appropriate for packings of ellipses in the plane or ellipsoids in space?

Even for a single ellipse in a triangle, what does a maximal ellipse with a fixed axis ratio look like?

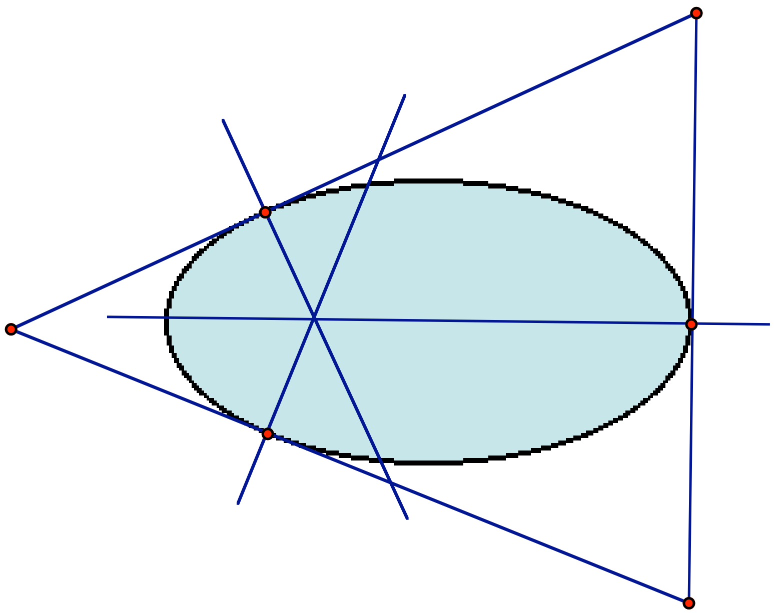
(Joint work with A. Donev, S. Torquato, F. Stillinger, Chaiken, et. al.)

An ellipse in a triangle



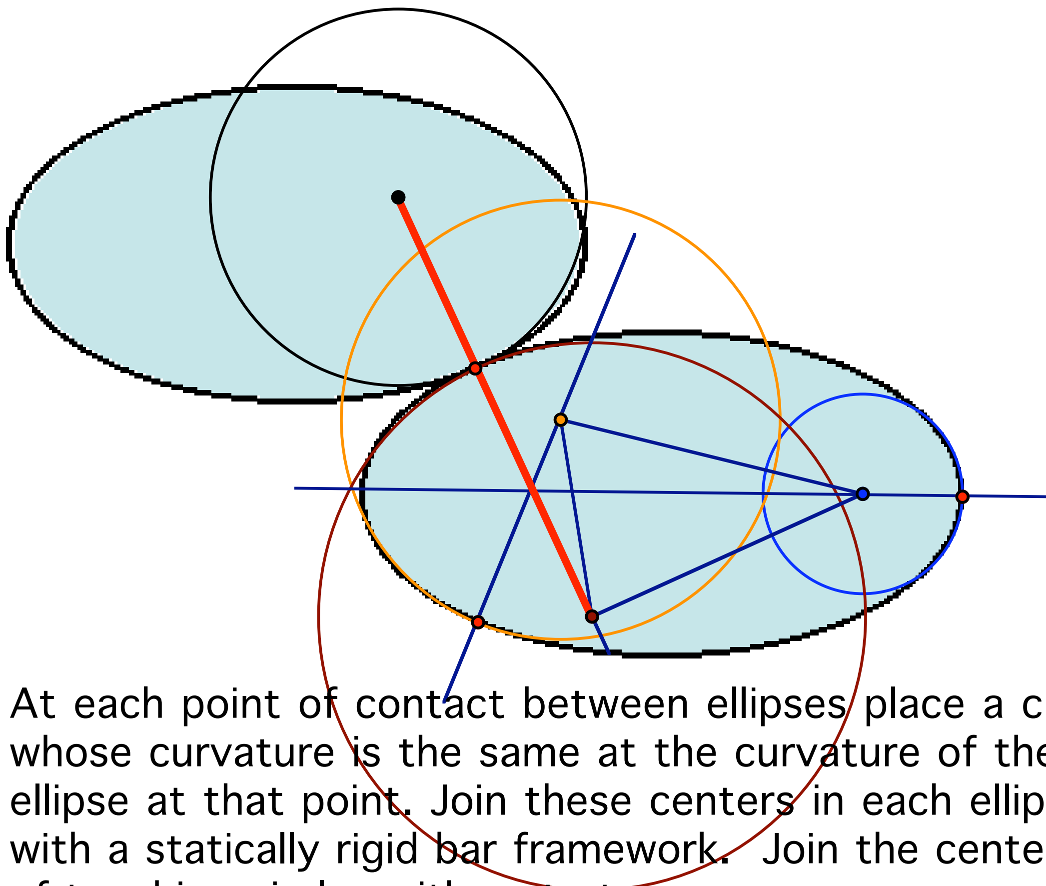
If the lines through the points of contact do not intersect in a point, then a rotation (counterclockwise here) about the triangle enclosed moves the ellipse into the triangle.

The critical case



When the 3 lines meet at a point there is an "infinitesimal flex" of the ellipse inside the triangle. So there is no occasion when the ellipse is "infinitesimally rigid" inside the triangle.

Modeling ellipses



At each point of contact between ellipses place a circle whose curvature is the same at the curvature of the ellipse at that point. Join these centers in each ellipse with a statically rigid bar framework. Join the centers of touching circles with a strut.

Consequences of the model

- The tensegrity paradigm applies to irregularly shaped particles.
- The notion of being infinitesimally rigid applies, BUT, unlike the case when all the particles are circles (or spheres in space), being rigid does not always imply infinitesimal rigidity. (The canonical push does not work in general.)
- If the structure is infinitesimally rigid, then there is a minimum number of contacts that are necessary. We calculate that next.

Contacts for ellipses

Suppose a packing of ellipses in the plane is such that all are pinned except for n which are allowed to move, and the system is infinitesimally rigid. Each ellipse has 3 degrees of freedom and each contact corresponds to 2 ellipses, except for the boundary. So if n is large enough so that the boundary effects are negligible, and if Z is the average degree of a jammed ellipse packing, then

$$Z \geq 6 - O(n^{-1/2}),$$

which seems unlikely for congruent ellipses.

Contact numbers for ellipsoids

Doing the same calculation in 3-space, where each ellipsoid has 3 distinct axis lengths, the degrees of freedom of each ellipsoid is 6. Then the average degree is $Z \geq 12 - O(n^{-2/3})$, which seems almost impossible for congruent ellipsoids unless they are a small perturbation of one of the standard most dense packings of congruent spheres.

Almost spherical ellipsoids

There was a famous argument between Newton and Gregory about how many spheres can touch a single sphere, all with the same radius in a packing. Newton said 12, Gregory 13. Newton was right as shown in the 1950's. This must also be true for ellipsoids if the axis ratios are close enough to 1. So the coordination number for such ellipsoids cannot be greater than 12 either. So if any sort of random packing of such packings is jammed, its coordination number should be closer to 6 than 12 and it cannot be statically rigid.

Contact numbers for spheroids

When the ellipsoid has one degree of rotational symmetry, i.e. when two of the three axes are the same length, that subtracts from the degrees of freedom of the ellipsoids. In that case, for infinitesimal rigidity, one needs $Z \geq 10 - O(n^{-2/3})$ for the coordination number.

Experimental results

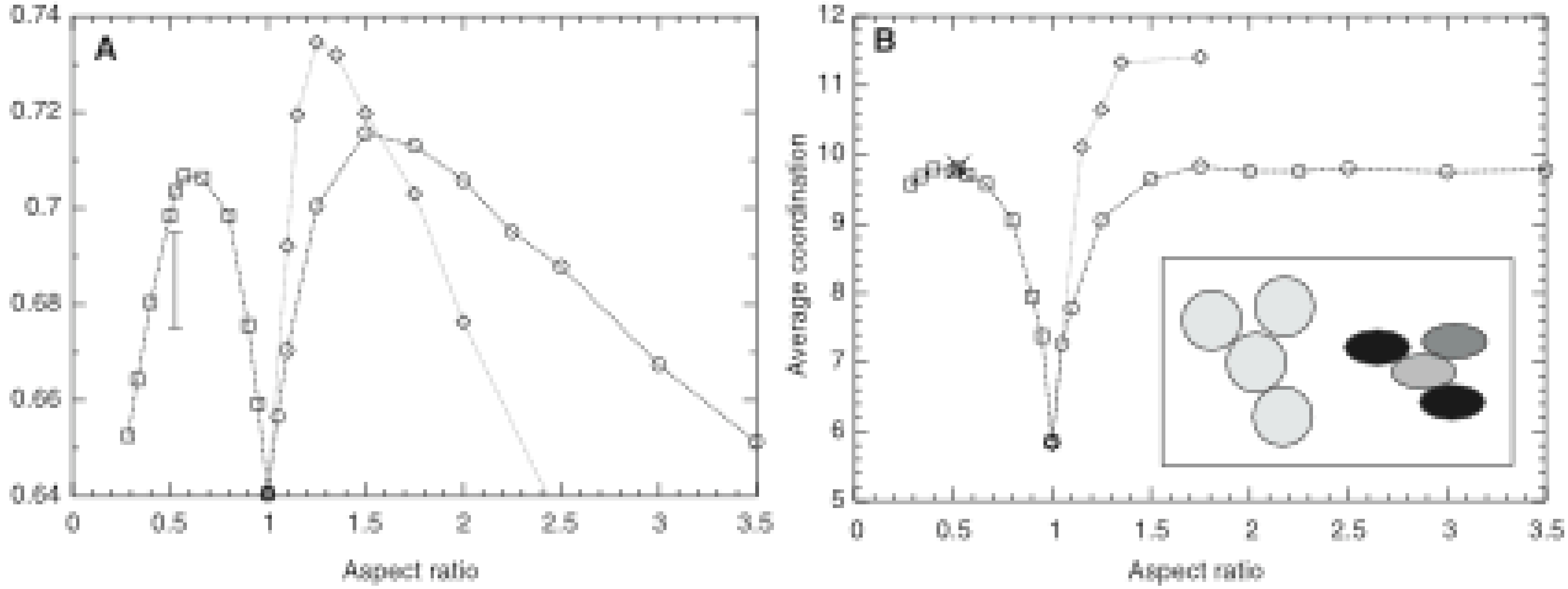
In the 8 author joint paper in Science, it is shown that in dimension 3 with congruent (monodispersed) ellipsoids with an axis ratio chosen in the range close to 1, but not equal to, have coordination number less than 12 in the case when all axes are different and less than 10 (about 9.5 for some) when two of the axes are the same.

The moral of the story

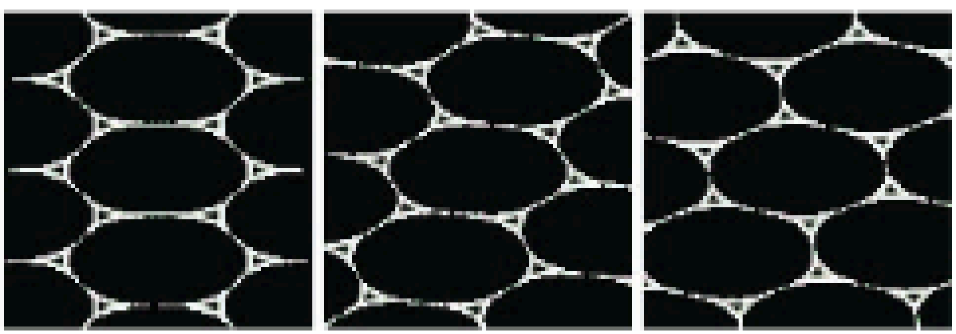
- Random, even mildly random, packings of ellipsoids must not be infinitesimally rigid. They must be prestress stable.
- “Random” packings of monodispersed ellipsoids with the appropriately chosen axis ratios tend to have surprisingly high packing densities.

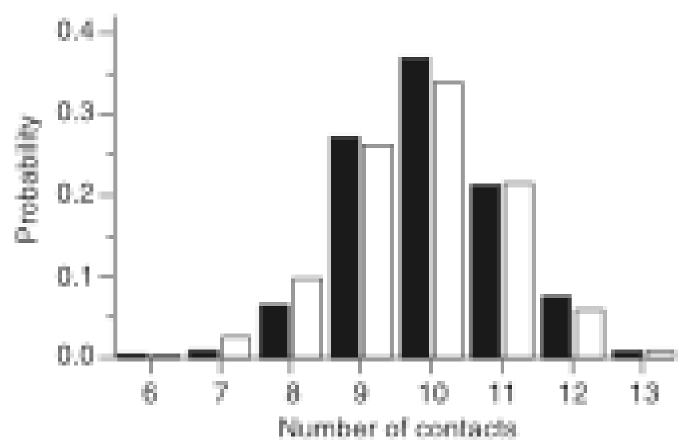
Packing densities and coordination numbers

Axis ratios	Packing density	Coordination number
1 : 1 : 1	0.63	6
1 : 1 : 1.5	0.70	9.5
1 : 1.3 : 0.77	0.73	11.4
1 : 1 : 0.6 (m&m's)	0.67	9.8



he experimental result for the regular candies (cross). Inset: Introducing asphericity makes a locally jammed particle free
ıge of neighbors.





. More contacts per
ıinate all local and

shows through a sequence of frames how one
can distort this collectively jammed packing

The m&m container and simulation

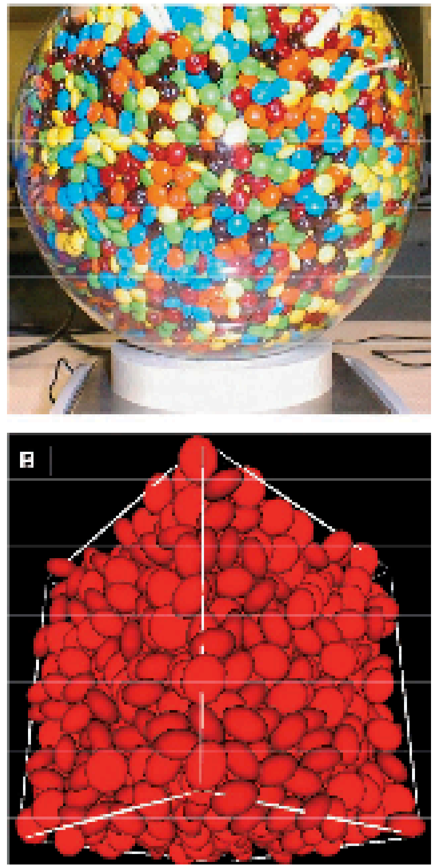


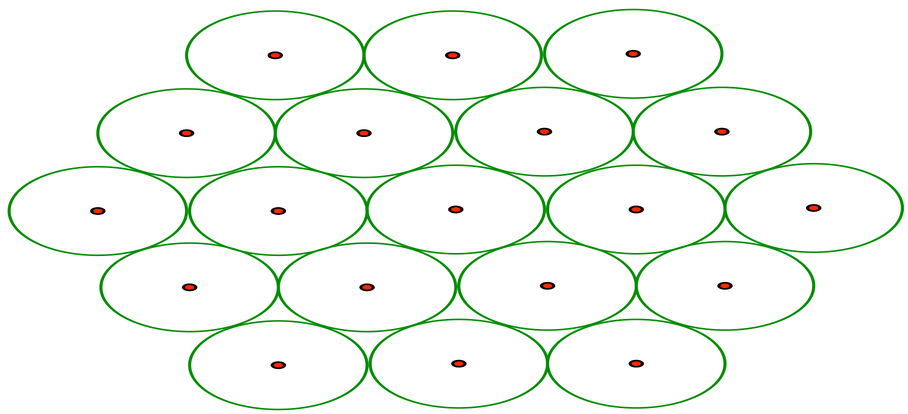
Fig. 1. (A) An experimental packing of the regular candies. (B) Computer-generated packing of 1000 oblate ellipsoids with $\alpha = 1.9:1$.

Remarks

Theorem (L. Fejes Toth 1940's): The most dense packing of any centrally symmetric convex body in the plane is achieved with a lattice packing.

In particular, for any ellipse, it has maximum packing density $\pi/\sqrt{12}=0.90699\ldots$, the same as the most dense packing of congruent circles, and for a periodic packing it is achieved by a one parameter family of ellipses obtained by some rotation of the standard circle packing dilated in one direction only.

An ellipse packing with max density



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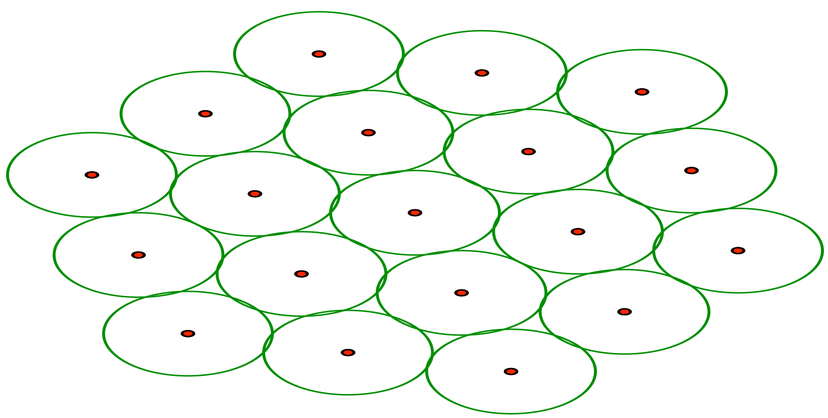

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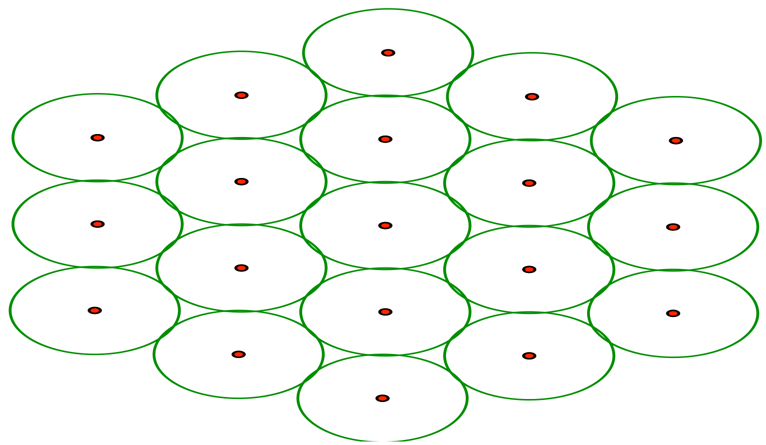
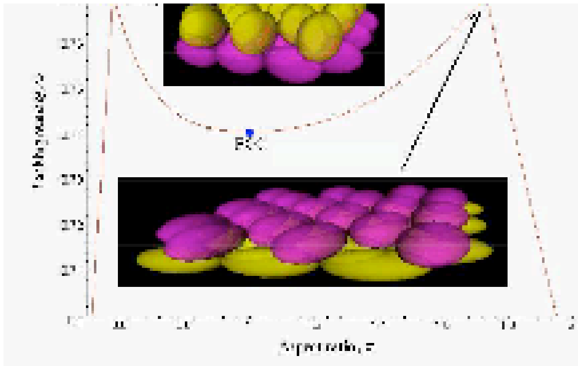


Packing congruent ellipsoids

A. Bezdek and W. Kuperberg showed that there are packings of congruent ellipsoids whose packing density exceeds $\pi/\sqrt{18}=0.7408\ldots$, the most dense packing of congruent spheres in 3-space. This was improved by Donev, Chaiken, Stillinger, Torquato to 0.7704…



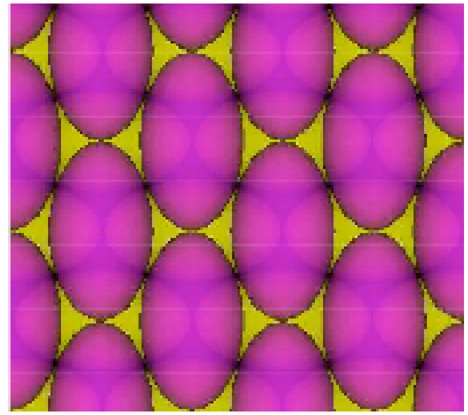


Figure [illegible]. The density of the laminated crystal packing of ellipsoids as a function of the aspect ratio α ($\beta = 1$). The point $\alpha = 1$ corresponding to the FCC lattice sphere packing is shown, along with the two sharp maxima in the density for prolate ellipsoids with $\alpha = \sqrt{3}$ and oblate ellipsoids with $\alpha = 1/\sqrt{3}$, as illustrated in the insets.

The ellipsoid packing

Each layer is as in the previous slide. But each layer is turned 90° and laid down onto the next layer. It then fits better than laying them down in a parallel manner. The axis ratio of the ellipsoids in each layer is $\sqrt{3}$. The length in the third direction is arbitrary. So this 77% density can be achieved when any of the 3 axis ratios is $\sqrt{3}$. Each ellipsoid is in contact with 14 others.

Note that the most dense random arrangement of ellipsoids was one where one of the axis ratios way 1.68, where as $\sqrt{3}=1.732$….

Question

Is there an upper bound □ < 1 for the best packing density of congruent ellipsoids, *independent of the axis ratios*?