Math 762 Homework Assignment, Due Thursday, March 8

1. With this exercise you will prove that any distance-preserving transformation of \( \mathbb{E}^d \) is given by an orthogonal transformation followed by a translation. We call such a transformation a \textit{congruence} of \( \mathbb{E}^d \). Let \( f : \mathbb{E}^d \to \mathbb{E}^d \) be any function such that for all \( p_1, p_2 \) in \( \mathbb{E}^d \), \( |f(p_1) - f(p_2)|^2 = |p_1 - p_2|^2 \).

   a. Assume \( f(0) = 0 \). Show that for any \( p_1, p_2 \) in \( \mathbb{E}^d \), \( f(p_1) \cdot f(p_2) = p_1 \cdot p_2 \). (Hint: Use the vector ‘polarization’ identity \( p_1 \cdot p_2 = (p_1^2 + p_2^2 - |p_1 - p_2|^2)/2 \).

   b. Assume \( f(0) = 0 \). Use part a.) to show that \( f \) is linear. Show that for any \( p_1, p_2 \) in \( \mathbb{E}^d \), \( |f(\alpha p_1 + p_2) - \alpha f(p_1) - f(p_2)|^2 = 0 \), where \( \alpha \) is an arbitrary scalar.

   c. Use part a.) and part b.) to show that \( f(p) = Ap + b \), where \( A \) is a \( d \)-by-\( d \) orthogonal matrix, and \( b \) is a vector in \( \mathbb{E}^d \).

2. Let \( p = (p_1, \ldots, p_n) \) and \( q = (q_1, \ldots, q_n) \) be two configurations of points in \( \mathbb{E}^d \), such that for all \( 1 \leq i < j \leq n \), \( |p_i - p_j| = |q_i - q_j| \). Show that there is a congruence \( f : \mathbb{E}^d \to \mathbb{E}^d \) such that for \( 1 \leq i \leq n \), \( f(p_i) = q_i \). Furthermore, if the affine span of \( p \) is \( d \)-dimensional, the congruence \( f \) is unique.

3. Suppose that \( p_1, p_2, p_3, p_4, p_5 \) are five points in Euclidean 3-space such that \( |p_i - p_{i+1}| \) is constant for \( i = 1, \ldots, 5 \) (indices \( \equiv \) mod 5), and the angles from \( p_{i-1} \) to \( p_i \) to \( p_{i+1} \) are equal to \( \theta \), constant for all \( i \). Let \( f : p \to p \) be the function defined by \( f(p_i) = p_{i+1} \), indices \( \equiv \) mod 5.

   a. Use problem 2 to show that \( f \) extends to a congruence of \( \mathbb{E}^3 \), and that this extension can be taken to be of order 5. In other words \( f \) composed with itself 5 times is the identity.

   b. Show that the congruence \( f \) of part a.) fixes the centroid \( (p_1+p_2+p_3+p_4+p_5)/5 \).

   c. Assume that the centroid of \( p \) of part b.) above is the origin, so the congruence of part a.) is a linear transformation given by an orthogonal matrix \( A \). Since \( A^5 = I \) by part a.), conclude that the determinant of \( A \) is 1. Thus \( A \) is a rotation about some line through the origin.

   d. Conclude that the affine span of \( p \) must be 2-dimensional.