BAIRE MEASURABLE MATCHINGS IN ACYCLIC LOCALLY FINITE BOREL GRAPHS

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ABSTRACT. We show that the class of graphs on topological spaces which have Baire measurable perfect matchings includes all acyclic locally finite Borel graphs on Polish spaces of degree at least three.

A graph on a set X is an irreflexive symmetric subset G of $X \times X$. A *matching* of G is a fixed-point-free involution ι of a subset of X whose graph is contained in G. A matching is *perfect* if its domain is X.

A *G-path* is a sequence $(x_i)_{i \in n+1}$ such that $(x_i, x_{i+1}) \in G$ for all $i \in n$. We refer to n as the *length* of the G-path. A graph G is *acyclic* if there is at most one injective G-path between any two points.

The *degree* of a point y is given by $\deg_G(y) = |\{x \in X \mid (x, y) \in G\}|$. A graph has *degree n* if every point has degree n. We say that a graph has *degree at least n* if every point has degree at least n. A graph is *locally finite* if every point has finite degree.

It is well known that there are acyclic Borel graphs of degree two which do not have Baire measurable perfect matchings. It is also known that there are acyclic Borel graphs of countably infinite degree which do not have Baire measurable perfect matchings. Our main result here rules out the analogous result for graphs of intermediate degree.

Let E_G^X denote the smallest equivalence relation on X containing G. The *connected component* of a point is its E_G^X-class. A set $Y \subseteq X$ is *G-invariant* if it is the union of connected components. The *restriction* of G to a set $Y \subseteq X$ is the graph on Y given by $G \upharpoonright Y = G \cap (Y \times Y)$.

A *G-ray* is a sequence $(x_i)_{i \in \omega}$ such that $(x_i, x_{i+1}) \in G$ for all $i \in \omega$. We say that a sequence $(x_i)_{i \in I}$ has *low even degree* if $\deg_G(x_i) = 2$ for all even natural numbers $i \in I$.

Theorem 1 (ZF). *Suppose that X is a Polish space, G is an acyclic locally finite Borel graph on X of degree at least two, and there is no injective G-ray of low even degree. Then there is a Borel matching of G whose domain is a comeager G-invariant Borel set.*

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Proof. Let $\mathscr{X}$ denote the set of pairs (S, T) of finite subsets of X such that $S \subseteq T$ and $T \times T \subseteq E_G^X$, equipped with the Borel structure it inherits from X, and let $\mathscr{G}$ denote the graph on $\mathscr{X}$ given by

$$\mathscr{G} = \{((S,T),(S',T')) \in \mathscr{X} \times \mathscr{X} \mid (S,T) \neq (S',T') \text{ and } T \cap T' \neq \emptyset\}.$$

Proposition 4.1 of [Mil08] yields a Borel coloring $c \colon \mathscr{X} \to \omega$ of $\mathscr{G}$.

We will now define a decreasing sequence $(X_s)_{s \in \omega^{<\omega}}$ of Borel subsets of X such that for all $s \in \omega^{<\omega}$ the graph $G \restriction X_s$ has degree at least two and no injective $(G \restriction X_s)$-ray has low even degree. We will simultaneously produce an increasing sequence $(\iota_s)_{s \in \omega^{<\omega}}$ of Borel matchings of G such that for all $s \in \omega^{<\omega}$ the domain of ι_s is $X \setminus X_s$.

We begin by setting $X_\emptyset = X$ and $\iota_\emptyset = \emptyset$. Suppose now that we have already defined X_s and ι_s. The *G-boundary* of a set $Y \subseteq X$ is given by $\partial_G(Y) = \{y \in Y \mid \exists x \in X \setminus Y \ ((x,y) \in G)\}$. Let $\mathscr{X}_s$ denote the set of pairs $(S,T) \in \mathscr{X}$ which satisfy the following conditions:

1. The inclusion $\partial_{G \restriction X_s}(X_s \setminus S) \subseteq T \subseteq X_s$ holds.
2. The graph $G \restriction (X_s \setminus S)$ has degree at least two.
3. No injective $G \restriction (X_s \setminus S)$-path passing through both a point in $\partial_{G \restriction X_s}(X_s \setminus S)$ and a point in $\partial_{G \restriction X_s}(T)$ has low even degree.
4. There is a perfect matching of $G \restriction S$.

For each $i \in \omega$, define $X_{s^\frown i} \subseteq X_s$ by

$$X_{s^\frown i} = X_s \setminus \bigcup \{S \mid \exists T \ ((S,T) \in \mathscr{X}_s \text{ and } c(S,T) = i)\}.$$

Lemma 2. *Suppose that $i \in \omega$ and $s \in \omega^{<\omega}$. Then $G \restriction X_{s^\frown i}$ has degree at least two.*

Proof of lemma. Suppose that $x \in X_{s^\frown i}$. If $x \notin \partial_{G \restriction X_s}(X_{s^\frown i})$, then $\deg_{G \restriction X_{s^\frown i}}(x) = \deg_{G \restriction X_s}(x) \geq 2$. Otherwise, there exists $(S,T) \in \mathscr{X}_s$ such that $c(S,T) = i$ and $x \in \partial_{G \restriction X_s}(X_s \setminus S)$. Condition (1) implies that $x \in T$, so (S,T) is the unique such pair, thus condition (2) ensures that $\deg_{G \restriction X_{s^\frown i}}(x) = \deg_{G \restriction (X_s \setminus S)}(x) \geq 2$, and it follows that $G \restriction X_{s^\frown i}$ has degree at least two. ⊠

Lemma 3. *Suppose that $i \in \omega$ and $s \in \omega^{<\omega}$. Then there is no injective $(G \restriction X_{s^\frown i})$-ray of low even degree.*

Proof of lemma. If $(x_k)_{k \in \omega}$ is an injective $(G \restriction X_{s^\frown i})$-ray of low even degree, then condition (3) ensures that $x_k \notin \partial_{G \restriction X_s}(X_{s^\frown i})$ for all $k \in \omega$, thus $(x_k)_{k \in \omega}$ is a $(G \restriction X_s)$-ray of low even degree, a contradiction. ⊠

Condition (4) ensures that ι_s extends to a Borel matching $\iota_{s^\frown i}$ of G with domain $X \setminus X_{s^\frown i}$.

For each $p \in \omega^\omega$, set $X_p = \bigcap_{n \in \omega} X_{p \restriction n}$ and $\iota_p = \lim_{n \to \omega} \iota_{p \restriction n}$. Clearly ι_p is a Borel matching of G with domain $X \setminus X_p$. It remains to show that there exists $p \in \omega^\omega$ such that $[X_p]_{E^X_G}$ is meager. We will in fact show that $\forall^* p \in \omega^\omega \forall^* x \in X \ ([x]_{E^X_G} \cap X_p = \emptyset)$. By the Kuratowski-Ulam Theorem (see, for example, Theorem 8.41 of [Kec95]), it is enough to show that $\forall x \in X \forall^* p \in \omega^\omega \ (x \notin X_p)$. For this, it is enough to show that $\forall s \in \omega^{<\omega} \forall x \in X_s \exists i \in \omega \ (x \notin X_{s \frown i})$.

Towards this end, suppose that $s \in \omega^{<\omega}$ and $x \in X_s$. Then there exists $y \in X_s$ such that $(x, y) \in G$.

Lemma 4. *There is a finite set $S \subseteq X_s$ with the property that $x, y \in S$, $G \restriction (X_s \setminus S)$ has degree at least two, and $G \restriction S$ has a perfect matching.*

Proof of lemma. A set $Y \subseteq X$ is *G-connected* if the graph $G \restriction Y$ has at most one connected component. We will recursively construct increasing sequences $(S_k)_{k \in \omega}$ of finite G-connected sets containing x and y and $(\iota_k)_{k \in \omega}$ of matchings of G such that the domain of ι_k is S_k for all $k \in \omega$. We begin by setting $S_0 = \{x, y\}$ and $\iota_0 = (x\ y)$. Given S_k and ι_k, observe that for each connected component C of $G \restriction (X_s \setminus S_k)$, there is at most one point $x \in C$ such that $|C \cap G_x| = 1$. Let ι_{k+1} denote the minimal extension of ι_k to an involution which associates every such x with the unique element of $C \cap G_x$, and let S_{k+1} denote the domain of ι_{k+1}. This completes the recursive construction.

Set $S = \bigcup_{k \in \omega} S_k$ and $\iota = \lim_{k \to \omega} \iota_k$. It is clear that $x, y \in S$, $G \restriction (X_s \setminus S)$ has degree at least two, and ι is a perfect matching of $G \restriction S$, so it only remains to show that S is finite. If this is not the case, then we can recursively construct pairs $(x_{2k}, x_{2k+1}) \in \mathrm{graph}(\iota_{k+1})$ with $(x_{2k+1}, x_{2k+2}) \in G$ by ensuring that the intersection of S with the connected component of x_{2k} in $G \restriction (X_s \setminus S_k)$ is infinite. Then $(x_k)_{k \in \omega}$ is an injective $(G \restriction X_s)$-ray of low even degree, a contradiction. ⊠

Lemma 5. *There is a finite set $T \subseteq X_s$ with $S \cup \partial_{G \restriction X_s}(X_s \setminus S) \subseteq T$ such that no injective $G \restriction (X_s \setminus S)$-path passing through both a point in $\partial_{G \restriction X_s}(X_s \setminus S)$ and a point in $\partial_{G \restriction X_s}(T)$ has low even degree.*

Proof of lemma. It is enough to show that for all $x \in X_s \setminus S$ there exists $n \in \omega$ such that no injective $G \restriction (X_s \setminus S)$-path beginning at x has length n and low even degree. Towards this end, simply note that if there are arbitrarily long injective $G \restriction (X_s \setminus S)$-paths of low even degree beginning at some point x_0, then we can recursively choose x_n such that there are arbitrarily long injective $G \restriction (X_s \setminus S)$-paths of low even degree extending $(x_k)_{k \in n+1}$, in which case $(x_k)_{k \in \omega}$ is an injective $G \restriction (X_s \setminus S)$-ray of low even degree, a contradiction. ⊠

As $(S, T) \in \mathscr{X}_s$, it follows that $i = c(S, T)$ is as desired. ⊠

Theorem 6 (ZFC)**.** *Suppose that X is a Polish space, G is an acyclic locally finite Borel graph on X of degree at least two, and there is no injective G-ray of low even degree. Then G has a Baire measurable perfect matching.*

Proof. By Theorem 1, there is a Borel matching ι of G whose domain is a comeager G-invariant Borel set. The axiom of choice ensures that ι extends to a perfect matching of G, and it is clear that every such extension is Baire measurable. ⊠

A set $B \subseteq X$ is *ω-universally Baire* if for every continuous function $\varphi \colon \omega^\omega \to X$ the set $\varphi^{-1}(B)$ has the Baire property.

Theorem 7 ($\mathtt{ZFC} + \mathtt{add}(\mathtt{meager}) = \mathfrak{c}$)**.** *Suppose that X is a Polish space, G is an acyclic locally finite Borel graph on X of degree at least two, and there is no injective G-ray of low even degree. Then G has an ω-universally Baire measurable perfect matching.*

Proof. Fix enumerations $(x_\alpha)_{\alpha\in\mathfrak{c}}$ of X and $(\varphi_\alpha)_{\alpha\in\mathfrak{c}}$ of the set of all continuous functions from ω^ω to X. The proof of Theorem 1 ensures that for all $\alpha \in \mathfrak{c}$ there is a Borel matching ι_α of G whose domain C_α is a G-invariant Borel set for which $x_\alpha \in C_\alpha$ and $\varphi_\alpha^{-1}(C_\alpha)$ is comeager. Set $D_\alpha = C_\alpha \setminus \bigcup_{\beta\in\alpha} C_\beta$ for all $\alpha \in \mathfrak{c}$, and observe that the function $\iota = \bigcup_{\alpha\in\mathfrak{c}} \iota_\alpha \restriction D_\alpha$ is as desired. ⊠

References

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