DISTANCE FROM MARKER SEQUENCES IN LOCALLY FINITE BOREL GRAPHS

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1. Introduction

The investigation of structure in marker sequences has been a recurring theme of the study of countable Borel equivalence relations and Borel graphs. Suppose Γ is a finitely generated group which acts on the space 2^Γ via the left shift action. Let $\mathrm{Free}(2^\Gamma)$ be the set of $x \in 2^\Gamma$ such that for all nonidentity $\gamma \in \Gamma$ we have $\gamma \cdot x \neq x$, and let $G(\Gamma, 2)$ be the graph on $\mathrm{Free}(2^\Gamma)$ where $x, y \in \mathrm{Free}(2^\Gamma)$ are adjacent if there is a generator γ of Γ such that $\gamma \cdot x = y$. Let $d_{G(\Gamma,2)}$ be the graph distance metric for $G(\Gamma, 2)$. A recent result of Gao, Jackson, and Seward states the following.

Theorem 1.1 (Gao, Jackson, and Seward)**.** *Suppose Γ is a finitely generated infinite group and $f\colon \mathbb{N} \to \mathbb{N}$ tends to infinity. Then for every Borel marker sequence $\{A_n\}_{n\in\mathbb{N}}$ for $G(\Gamma, 2)$, there exists an $x \in \mathrm{Free}(2^\Gamma)$ such that for infinitely many n, we have $d_{G(\Gamma,2)}(x, A_n) < f(n)$*

This result led us to ask the following question: what can we say if the function $f\colon \mathbb{N} \to \mathbb{N}$ is allowed to vary depending on the point x? Of course, we cannot possibly draw an analogous conclusion for an arbitrary Borel way of associating some $f_x\colon \mathbb{N} \to \mathbb{N}$ to each point x in our space; given a Borel marker sequence $\{A_n\}_{n\in\mathbb{N}}$ for a graph G on X, we could define $f_x(n) = d_G(x, A_n)$ for all $x \in X$. Instead, we show the existence of some Borel map $x \mapsto f_x$ for which we can draw a stronger conclusion than that of Theorem 1.1, showing closeness for all n instead of just infinitely many n. This is true even when we generalize to arbitrary locally finite non-smooth graphs.

Theorem 1.2. *Suppose G is a locally finite non-smooth Borel graph on X. Then there exists a Borel map associating to each $x \in X$ a function $f_x\colon \mathbb{N} \to \mathbb{N}$ such that for every Borel marker sequence $\{A_n\}_{n\in\mathbb{N}}$ for G, there exists an $x \in X$ such that for all n, we have $d_G(x, A_n) < f_x(n)$.*

Now when G is smooth, an easy diagonalization constructs marker sequences that do not satisfy the conclusion of our theorem. Hence, this result provides a novel way of characterizing smoothness; a locally finite Borel graph is smooth if and only if it does not admit Borel marker sequences that are somewhere "far" from every point.

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We remark here that in Theorem 1.2, the map $x \mapsto f_x$ may always be chosen so that it is a Borel homomorphism from the equivalence relation graphed by G to tail equivalence on $\mathbb{N}^{\mathbb{N}}$. We note that standard "sparse" Borel marker sequence constructions show that a map $x \mapsto f_x$ witnessing Theorem 1.2 cannot take a constant value on each connected component of G.

The theorem is proved in two steps. First, we use the fact that all Borel subsets of $[\mathbb{N}]^{\mathbb{N}}$ are completely Ramsey to give an example of a Borel graph G satisfying the conclusion of Theorem 1.2. Then, we conclude the full result using the Glimm-Effros dichotomy. We show in the last section that this theorem cannot be proven using measure or category arguments.

2. Distance from marker sequences and the Ramsey property

Let $[\mathbb{N}]^{\mathbb{N}}$ be Ramsey space, the space of infinite subsets of $\mathbb{N}$. Given a finite set $s \subseteq \mathbb{N}$ and an infinite set $x \subseteq \mathbb{N}$ with $\min(x) > \max(s)$, recall the definition $[s,x]^{\mathbb{N}} = \{y \in [\mathbb{N}]^{\mathbb{N}} : s \subseteq y \subseteq s \cup x\}$. We can identify $[\mathbb{N}]^{\mathbb{N}}$ with a subset of $2^{\mathbb{N}}$ via characteristic functions, and we use the resulting subspace topology on $[\mathbb{N}]^{\mathbb{N}}$ throughout. A theorem of Galvin and Prikry [1] states that for every $[s,x]^{\mathbb{N}}$ and every Borel subset $B \subseteq [s,x]^{\mathbb{N}}$, there exists some $[t,y]^{\mathbb{N}} \subseteq [s,x]^{\mathbb{N}}$ such that either $[t,y]^{\mathbb{N}} \subseteq B$ or $[t,y]^{\mathbb{N}} \cap B = \emptyset$. From this, it is easy to see that the following:

Lemma 2.1 (Galvin-Prikry [1])**.** *If $\{B_n\}_{n\in\mathbb{N}}$ is a Borel partition of $[s,x]^{\mathbb{N}}$, then there exists some $n \in N$ and $[t,y]^{\mathbb{N}} \subseteq [s,x]^{\mathbb{N}}$ such that $[t,y]^{\mathbb{N}} \subseteq B_n$.*

Proof. Suppose not. Then we may construct a decreasing sequence $[s,x]^{\mathbb{N}} \supseteq [s_0,x_0]^{\mathbb{N}} \supseteq [s_1,x_1]^{\mathbb{N}} \supseteq \ldots$, where $[s_n,x_n]^{\mathbb{N}} \cap B_n = \emptyset$, and s_n has at least n elements. But then setting $z = \bigcup_n s_n$, we see that $z \in [s,x]$, and $z \notin B_n$ for all n, hence $\{B_n\}_{n\in\mathbb{N}}$ does not partition $[s,x]^{\mathbb{N}}$. □

The *odometer* $\sigma\colon [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ is defined via the identification of $[\mathbb{N}]^{\mathbb{N}}$ as a subspace of $2^{\mathbb{N}}$ by setting $\sigma(x) = 0^n1y$ if $x = 1^n0y$, and fixing $\sigma(111\ldots) = 111\ldots$ on the sequence consisting of all ones. Define also $\tau\colon [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ by setting $\tau(x) = \{n-1 : n \in x \wedge n > 0\}$. Let G_t be the graph on $[\mathbb{N}]^{\mathbb{N}}$ generated by these two functions, where $x, y \in [\mathbb{N}]^{\mathbb{N}}$ are adjacent if either $\sigma(x) = y$, $\sigma(y) = x$, $\tau(x) = y$, or $\tau(y) = x$. So G_t is graphing of tail equivalence on $[\mathbb{N}]^{\mathbb{N}}$, and every vertex in G_t has degree ≤ 5.

Lemma 2.2. *Consider the graph G_t defined on $[\mathbb{N}]^{\mathbb{N}}$ as above, and for each $x \in [\mathbb{N}]^{\mathbb{N}}$, let $f_x(n)$ be equal to the nth element of x. Then for every Borel marker sequence $\{A_n\}_{n\in\mathbb{N}}$ for G_t, there is an $x \in [\mathbb{N}]^{\mathbb{N}}$ such that for all n, we have $d_{G_t}(x, A_n) < f_x(n)$.*

Proof. We construct x as the intersection of a decreasing sequence $[s_0,x_0]^{\mathbb{N}} \supseteq [s_1,x_1]^{\mathbb{N}} \supseteq \ldots$, where s_n has exactly n elements. Let $s_0 = \emptyset$, and $x_0 = \mathbb{N}$. Now given $[s_n,x_n]^{\mathbb{N}}$, since the sets $\{\{y \in [s_n,x_n]^{\mathbb{N}} : d(y, A_n) = k\}\}_{k\in\mathbb{N}}$ partition $[s_n,x_n]$, we may apply Lemma 2.1 to obtain some $[t,y] \subseteq [s_n,x_n]^{\mathbb{N}}$ and some k such that every element of $[t, y]^{\mathbb{N}}$ is distance exactly k from A_n. Since $s_n \subseteq t$, there is some m such that m applications of the odometer applied to $[s_n, y]^{\mathbb{N}}$ yield $\sigma^m([s_n, y]^{\mathbb{N}}) = [t, y]^{\mathbb{N}}$. Hence by the triangle inequality, we see that there is some $k^* = k + m$ such that all the elements of $[s_n, y]$ are distance $\leq k^*$ from A_n. Now let l be the least element of y that is strictly greater than k^* and $\max(s_n)$, let $s_{n+1} = s_n \cup \{l\}$, and $x_{n+1} = y \setminus \{0, 1, \ldots, l\}$. We have ensured then that every element of $[s_{n+1}, x_{n+1}]^{\mathbb{N}}$ has distance $\leq l$ from A_n. □

We remark here that the above proof works equally well for the usual graphing of E_0 on $[\mathbb{N}]^{\mathbb{N}}$ induced by the odometer. We have used the larger graph G_t because we will need a locally finite graphing of tail equivalence with our desired property in order to finish the proof of Theorem 1.2.

We need one more easy lemma before we complete the theorem. The lemma roughly states that this question of closeness to marker sequences is independent of the particular locally finite Borel graph we choose, and depends only on the equivalence relation we have graphed.

Given a Borel graph G on X, and a Borel map $x \mapsto f_x$ from $X \to \mathbb{N}^{\mathbb{N}}$, say that a marker sequence $\{A_n\}_{n\in\mathbb{N}}$ *satisfies* $x \mapsto f_x$ *for* G if for all $x \in X$ there exists an n such that $d_G(x, A_n) \geq f_x(n)$.

Lemma 2.3. *Suppose G and H are locally finite Borel graphs on a standard Borel space X having the same connected components. Then for every Borel map $x \mapsto g_x$ from $X \to \mathbb{N}^{\mathbb{N}}$, there exists a Borel map $x \mapsto h_x$ such that for every marker sequence $\{A_n\}_{n\in\mathbb{N}}$, if $\{A_n\}_{n\in\mathbb{N}}$ satisfies $x \mapsto h_x$ for H, then $\{A_n\}_{n\in\mathbb{N}}$ satisfies $x \mapsto g_x$ for G.*

Proof. Since the graphs are locally finite, there are only finitely many points a fixed distance from each $x \in X$. Hence, we may define $h_x(n)$ to be the least k such that $d_G(x, y) \leq k$ for all $y \in X$ such that $d_G(x, y) \leq g_x(n)$. □

We now complete the proof of Theorem 1.2.

Theorem 1.2. Suppose G is a locally finite non-smooth Borel graph on X. Then there exists a Borel map associating to each $x \in X$ a function $f_x \colon \mathbb{N} \to \mathbb{N}$ such that for every Borel marker sequence $\{A_n\}_{n\in\mathbb{N}}$ for G, there exists an $x \in X$ such that for all n, we have $d_G(x, A_n) < f_x(n)$.

Proof of Theorem 1.2. Suppose G is not smooth. Let E be the equivalence relation graphed by G, and E_t be the equivalence relation of tail equivalence on $[\mathbb{N}]^{\mathbb{N}}$. By the Glimm-Effros dichotomy, there must be some E-invariant Borel set A such that $E \restriction A \cong_B E_t$. But then $G \restriction A$ and the graph G_t from Lemma 2.2 are two different locally finite Borel graphings of the same equivalence relation. Hence, by Lemma 2.3, we can find a Borel $x \mapsto h_x$ from $A \to \mathbb{N}^{\mathbb{N}}$ so that no Borel marker sequence can satisfy $G \restriction A$ for $x \mapsto h_x$. Hence, any Borel extension of $x \mapsto h_x$ to a function $x \mapsto f_x$ defined on X suffices to prove the theorem. □

3. Measure and category

In this section, we prove the following:

Proposition 3.1. *Suppose G is a locally finite Borel graph on X, and $x \mapsto f_x$ is a Borel map from $X \to \mathbb{N}^{\mathbb{N}}$. Then*

1. *For every Borel probability measure μ on X, there is a G-invariant μ-conull set B and a Borel marker sequence $\{A_n\}_{n \in \mathbb{N}}$ for $G \restriction B$ such that for every $x \in X$, there is an n such that $d_G(x, A_n) \geq f_x(n)$.*
2. *For every compatible Polish topology τ on X, there is a G-invariant τ-comeager set B and a Borel marker sequence $\{A_n\}_{n \in \mathbb{N}}$ for $G \restriction B$ such that for every $x \in X$, there is an n such that $d_G(x, A_n) \geq f_x(n)$.*

Proof. Let $\{B_n\}_{n \in \mathbb{N}}$ be a Borel marker sequence for G, and let $C_{i,n} = \{x \in X : d_G(x, A_i) < f_x(n)\}$. Note since $\{B_n\}_{n \in \mathbb{N}}$ is a marker sequence, for each n, we have $\bigcap_i C_{i,n} = \emptyset$.

For part (1), we may assume as usual that μ is G-quasi-invariant. Observe that for each n, the μ-measure of the sets $C_{i,n}$ goes to 0. Hence, we may find a sequence $i_0, i_1, i_2, \ldots$ such that $\mu(C_{i_n,n}) \to 0$. Now choose our marker sequence to be $\{A_n\}_{n \in \mathbb{N}}$ where $A_n = B_{i_n}$. This marker sequence has the required property on the complement of the nullset $\bigcap_i C_{i_n,n}$.

Part (2) follows using a similar argument, since relative to any basic open set, the set $C_{i,n}$ can be comeager for only finitely many i. □

References

[1] Fred Galvin and Karel Prikry, *Borel sets and Ramsey's theorem*, J. Symbolic Logic **38** (1973), 193–198. MR0337630 (49 #2399)