FINITE MONOID-VALUED MEASURE ALGEBRAS

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We fix an abelian semigroup \( \langle S, + \rangle \). We say that \( S \) is positive if it contains no additive identity. For \( m, n \in \mathbb{N} \), an \( m \times n \) \( S \)-matrix is an \( m \times n \) matrix \( A = (a_{i,j}) \) whose entries are elements of \( S \). If \( A = (a_{i,j}) \) is an \( m \times n \) \( S \)-matrix, let

\[
\mathbf{r}_A = \left( \sum_{j<n} a_{0,j}, \sum_{j<n} a_{1,j}, \ldots, \sum_{j<n} a_{m-1,j} \right)
\]
denote its sequence of row sums, and let

\[
\mathbf{c}_A = \left( \sum_{i<m} a_{i,0}, \sum_{i<m} a_{i,1}, \ldots, \sum_{i<m} a_{i,n-1} \right)
\]
denote its sequence of column sums.

We say that \( S \) splits four ways if for every \( r_0, r_1, c_0, c_1 \in S \) with \( r_0 + r_1 = c_0 + c_1 \), there is a \( 2 \times 2 \) \( S \)-matrix \( A \) with \( \mathbf{r}_A = (r_0, r_1) \) and \( \mathbf{c}_A = (c_0, c_1) \).

**Example 1.** Suppose that \( \langle G, +, < \rangle \) is an abelian group with identity \( 0_G \) and a translation-invariant partial order. We use \( G^+ \) to denote the positive semigroup \( \{ g : 0_G < g \} \). Denoting by \( \exists^+, \forall^+ \) quantification over \( G^+ \), we say \( G^+ \) splits under sums if

\[
\forall^+ g_0, g_1 \forall^+ k < g_0 + g_1 \exists^+ h_0 < g_0 \exists^+ h_1 < g_1 \ (h_0 + h_1 = k).
\]

It is not hard to see that \( G^+ \) splits four ways if and only if it splits under sums.

**Example 2.** As a special case of Example 1, suppose that \( \langle G, +, < \rangle \) is an abelian group with a translation-invariant linear order. In this case, \( G^+ \) splits four ways if and only if \( G^+ \) has no \( < \)-minimal element.

**Example 3.** If \( \langle L, \wedge, \vee \rangle \) is a lattice, we may view it as an abelian semigroup under the operation \( \vee \). A semigroup arising in this fashion always splits four ways: suppose \( r_0, r_1, c_0, c_1 \in L \) with \( r_0 \vee r_1 = c_0 \vee c_1 \). Then the matrix

\[
\begin{pmatrix}
  r_0 \wedge c_0 & r_0 \wedge c_1 \\
  r_1 \wedge c_0 & r_1 \wedge c_1
\end{pmatrix}
\]

has the required row and column sums. Additionally, such a lattice is a positive semigroup if and only if it contains no bottommost element (e.g., the cofinite subsets of \( \mathbb{N} \)).
Lemma 4. Suppose that $S$ is an abelian semigroup that splits four ways. Suppose further that $m, n \in \mathbb{N}$ and $r = (r_0, \ldots, r_{m-1})$, $c = (c_0, \ldots, c_{n-1})$ are sequences of elements of $S$ with $\sum_{i<m} r_i = \sum_{j<n} c_j$. Then there exists an $m \times n$ $S$-matrix $A$ such that $r_A = r$ and $c_A = c$.

Proof. We proceed by induction on $m+n$. The lemma is trivial when either of $m, n$ is less than 2, and the case $m = n = 2$ is granted by the assumption that $S$ splits four ways. By interchanging rows and columns if necessary, we may assume $m > 2$.

Suppose that $r = (r_0, \ldots, r_{m-1})$ and $c = (c_0, \ldots, c_{n-1})$ are as in the statement of the lemma. By the inductive hypothesis, we know there exists a $2 \times n$ $S$-matrix

$$A = \begin{pmatrix} a_{0,0} & \cdots & a_{0,n-1} \\ a_{1,0} & \cdots & a_{1,n-1} \end{pmatrix}$$

with $r_A = (\sum_{i<m} r_i, r_{m-1})$ and $c_A = (c_0, \ldots, c_1)$. Again using the inductive hypothesis, there exists a $(m-1) \times n$ $S$-matrix

$$B = \begin{pmatrix} b_{0,0} & \cdots & b_{0,n-1} \\ \vdots & \ddots & \vdots \\ b_{m-2,0} & \cdots & b_{m-2,n-1} \end{pmatrix}$$

with $r_B = (r_0, \ldots, r_{m-2})$ and $c_B = (a_{0,0}, \ldots, a_{0,1})$. We then simply observe that the matrix

$$\begin{pmatrix} b_{0,0} & \cdots & b_{0,n-1} \\ \vdots & \ddots & \vdots \\ b_{m-2,0} & \cdots & b_{m-2,n-1} \\ a_{1,0} & \cdots & a_{1,n-1} \end{pmatrix}$$

has the required row and column sums. \hfill \Box

Remark 5. Lemma 4 remains true for nonabelian semigroups, with the same proof, provided that row and column sums are reinterpreted in the obvious way.

We say that a monoid $\langle G, + \rangle$ with identity $0_G$ is **nonnegative** if $G^+ = G \setminus \{0_G\}$ is a (positive) semigroup. Equivalently, if $g_0 + g_1 = 0_G$, then $g_0 = g_1 = 0_G$. We fix such a monoid.

We now turn our attention to the main focus of the paper, the class of naturally ordered finite measure algebras equipped with a measure taking values in $G$. Given a Boolean algebra $\langle B, \land, \lor, 0, 1 \rangle$, a **positive $G$-valued measure** on $B$ is a function $\mu : B \rightarrow G$ such that for all $b_0, b_1 \in B$:

1. $\mu(b_0) = 0_G \iff b_0 = 0_G$;
2. if $b_0 \land b_1 = 0$, then $\mu(b_0 \lor b_1) = \mu(b_0) + \mu(b_1)$.

Fix a positive element $g_1 \in G$. The class $\text{OMBA}_{G,g_1}$ consists of structures of the form $B = \langle B, \land, \lor, 0, 1, \mu_B, <_B \rangle$, where $\langle B, \land, \lor, 0, 1 \rangle$ is a finite Boolean algebra, $\mu_B : B \rightarrow G$ is a positive $G$-valued measure with $\mu_B(1) = g_1$, and $<_B$ is an order induced antilexicographically by an ordering of the atoms of $B$. 
Theorem 6. Suppose that $G$ is a countable, nonnegative abelian monoid such that $G^+$ splits four ways, and that $g_1$ is a positive element of $G$. Then the class $OMBA_{G,g_1}$ is a Fraïssé order class.

Proof. We prove only that $OMBA_{G,g_1}$ satisfies the AP, since the other properties are routinely verified (in particular, JEP follows from AP upon considering the \{0,1\} Boolean algebra). Towards this end, fix $B, C, D \in OMBA_{G,g_1}$ as well as embeddings $f : B \to C$ and $g : B \to D$. Our goal is to find some $E \in OMBA_{G,g_1}$ and embeddings $r : C \to E$ and $s : D \to E$ satisfying $r \circ f = s \circ g$.

Let $b_0 >_B \cdots >_B b_{l-1}$ list the atoms of $B$. For each $k < l$, let $c_{0,k} >_C \cdots >_C c_{m_k-1,k}$ list the atoms below $f(b_k)$ in $C$. Similarly, let $d_{0,k} >_D \cdots >_D d_{n_k-1,k}$ list the atoms below $g(b_k)$ in $D$. In particular,

$$\sum_{i < m_k} \mu_C(c_{i,k}) = \sum_{j < n_k} \mu_D(d_{j,k}) = \mu_B(b_k).$$

For each $k < l$, we define two sequences of positive elements of $G$ by

$$r_k = (\mu_C(c_{0,k}), \ldots, \mu_C(c_{m_k-1,k})) \quad \text{and} \quad c_k = (\mu_D(d_{0,k}), \ldots, \mu_D(d_{n_k-1,k})).$$

These sequences satisfy the hypotheses of Lemma 4, so we may find a $G^+$-matrix $A_k = (a_{i,j,k})$ with $r_{A_k} = r_k$ and $c_{A_k} = c_k$.

Intuitively, we identify the atoms of $B$ with the collection of these matrices, the atoms of $C$ with the rows of these matrices, and the atoms of $D$ with their columns. Towards that end, let $E$ be the Boolean algebra generated by some set of distinct atoms indexed as $\{e_{ij,k} : k < l, i < n_k, j < m_k\}$. Let $\mu_E$ be the unique positive $G$-valued measure on $E$ such that for all $i,j,k$, $\mu_E(e_{ij,k}) = a_{i,j,k}$; such a measure exists by the nonnegativity of $G$.

We define embeddings $r : C \to E$ and $s : D \to E$ as the unique maps satisfying

$$r(c_{i,k}) = \bigvee_j e_{ij,k} \quad \text{and} \quad s(d_{j,k}) = \bigvee_i e_{ij,k}.$$  

Certainly

$$\mu_E(r(c_{i,k})) = \sum_j \mu_E(e_{ij,k}) = \sum_j a_{i,j,k} = \mu_C(c_{i,k}) \quad \text{and} \quad \mu_E(s(d_{j,k})) = \sum_i \mu_E(e_{ij,k}) = \sum_i a_{i,j,k} = \mu_D(d_{j,k}).$$

by the conditions on the row and column sums of the $G^+$-matrices $A_k$. Furthermore, for all $k < l$,

$$r \circ f(b_k) = s \circ g(b_k) = \bigvee_{i,j} e_{i,j,k}$$

so $r \circ f = s \circ g$. To complete the proof of AP, it remains only to define an ordering of the atoms of $E$ so that $r$ and $s$ preserve the orders of the atoms of $C$ and $D$.

We desire to order the union of the sets of leading atoms $X = \{e_{i0,k} : k < l \text{ and } i < m_k\}$ and $Y = \{e_{0jk} : k < l \text{ and } j < n_k\}$ in a way that induces an order
compatible with the orders $<_C$ and $<_D$. Once we have ordered the leading atoms,
we may order the remaining atoms however we like, so long as they are smaller than
the leading atoms.

Let $X$ be ordered by $e_{0k} <_X e_{0k'} \iff c_{i,k} <_C c_{i,k'}$. Similarly, let $Y$ be ordered
by $e_{0j} <_Y e_{0j'} \iff d_{j,k} <_D d_{j,k'}$. Notice that these two orderings coincide on
$X \cap Y = \{ e_{0k} : k < l \}$ since

\[
e_{0k} <_X e_{0k'} \iff c_{0,k} <_C c_{0,k'} \iff b_k <_B b_k' \iff d_{0,k} <_D d_{0,k'} \iff e_{00k} <_Y e_{00k'}.
\]

Thus, by the amalgamation property for finite linear orderings, there is an order on
$X \cup Y$ extending both $<_X$ and $<_Y$, so we have completed the proof. \(\square\)

**Remark 7.** Continuing the analysis of Example 2, the assumption that $G^+$ has
no minimal element is necessary. Indeed, suppose that $g$ is the minimal element
of $G^+$. Let $B = \langle B, \land, \lor, 0, 1, \mu_B, <_B \rangle$, where $B$ is the 4-element Boolean algebra
with atoms $\{ b_0, b_1 \}$, $\mu_B(b_i) = 2g$ for all $i < 2$, and $b_0 <_B b_1$. Let $C = \langle C, \land, \lor, 0, 1, \mu_C, <_C \rangle$ and $D = \langle D, \land, \lor, 0, 1, \mu_D, <_D \rangle$, where $C$ and $D$ both equal
the 16-element Boolean algebra with atoms $\{ a_0, a_1, a_2, a_3 \}$, $\mu_C(a_i) = \mu_D(a_i) = g$
for all $i < 4$. Finally, the orders are given by

\[
a_0 <_C a_1 <_C a_2 <_C a_3, \\
a_0 <_D a_2 <_D a_1 <_D a_3.
\]

Let $f : B \to C$ and $g : B \to D$ be the embeddings extending $f(b_0) = g(b_0) = a_0 \lor a_1$,
$f(b_1) = g(b_1) = a_2 \lor a_3$. A moment’s reflection reveals that the minimality of $g$
and the particular orders on $C$ and $D$ prevent the amalgamation of these structures.