EXAMPLES OF GENUS 5 CURVES

1. Genus 5 curves in $\mathbb{P}^2$

**Example 1.1.** A degree 5 plane curve with one node. Indeed, by the degree-genus formula,

$$p_g = \frac{(5 - 1)(5 - 2)}{2} - 1 = 5$$

A specific example of such a curve is $V(x^3yz + y^5 + z^5)$, with node at the point $[1 : 0 : 0]$.

**Proposition 1.2.** A curve $X$ of genus 5 can be represented as a plane quintic with one node iff it has a $g_3^1$.

*Proof.* Let $D$ be a $g_3^1$ which is effective. Then $\text{deg}(K_X - D) = 2 \cdot 5 - 2 - 3 = 5$. By Riemann-Roch, $h^0(D) - h^0(K_X - D) = 3 + 1 - 5 = -1$ and so $h^0(K_X - D) = 3$. It follows that $K_X - D$ is a $g_3^5$. One can use $|K_X - D|$ to map $X$ to $\mathbb{P}^2$, and the degree of the map is 5. Assume that the image only has nodes as singularities. By the degree-genus formula,

$$5 = p_g(X) = \frac{(5 - 1)(5 - 2)}{2} - \#\text{Nodes}$$

So there is only one node. Conversely, suppose that $X$ has a representation as a plane quintic curve with one node. Let $f : X \rightarrow \mathbb{P}^2$. Let $E$ be an effective divisor such that $O_X(E) \cong f^*O_{\mathbb{P}^2}(1)$. So $\text{deg}(E) = 5$. On the other hand, $\text{deg}(K_X - E) = 2 \cdot 5 - 2 - 5 = 3$. By Riemann-Roch, $h^0(E) - h^0(K_X - E) = 5 + 1 - 5 = 1$. Since $\text{Im}(f)$ is non degenerate, $h^0(E) = h^0(f^*O_{\mathbb{P}^2}(1)) \geq h^0(O_{\mathbb{P}^2}(1)) = 3$, which implies that $h^0(K_X - E) \geq 2$. So $E$ is special. By Clifford’s theorem,

$$h^0(E) - 1 \leq \frac{1}{2}\text{deg}(E) = \frac{5}{2}$$

$$h^0(E) \leq \frac{7}{2}$$

It follows that $h^0(E) = 3$ and $h^0(K_X - E) = 2$. Therefore $K_X - E$ is a $g_3^1$. □

**Example 1.3.** A degree 6 plane curve with 5 nodes.

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2. Genus 5 curves in $\mathbb{P}^4$

**Proposition 2.1.** Let $X$ be a curve of genus 5 and degree 8 in $\mathbb{P}^4$. Then $X$ must lie on 3 quadric surfaces.

**Proof.** Consider the short exact sequence of sheaves

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_X \to 0$$

Twisting it by 2 and taking the long exact sequence of sheaf cohomology, we have

$$0 \to H^0(\mathcal{I}_X(2)) \to H^0(\mathcal{O}_{\mathbb{P}^4}(2)) \to H^0(\mathcal{O}_X(2)) \to \cdots$$

By Riemann-Roch,

$$h^0(\mathcal{O}_X(2)) - h^0(\mathcal{O}_{\mathbb{P}^4}(2)) = 2 \cdot 8 + 1 - 5 = 12$$

Since $\deg(\mathcal{O}_X(-2)) < 0$, $h^0(\mathcal{O}_X(-2)) = 0$. Hence $h^0(\mathcal{O}_X(2)) = 12$. The long exact sequence implies that $h^0(\mathcal{I}_X(2)) \geq 3$. So $X$ lies on 3 quadric hypersurfaces. \(\square\)

**Example 2.2.** $X$ is a complete intersection of three nonsingular quadric hypersurfaces. It is of degree $2 \cdot 2 \cdot 2 = 8$. By Adjunction Formula, $\omega_X = \mathcal{O}_X(-5 + 2 + 2 + 2) = \mathcal{O}_X(1)$. $\deg K_X = 8 \cdot 1 = 2p_g - 2$, so $p_g = 5$.

**Proposition 2.3.** The linear system $|2\pi^*L - E|$ of $Bl_p\mathbb{P}^2$, which corresponds to the linear system of all conics passing through $p$ in $\mathbb{P}^2$, is very ample, and gives an embedding to $\mathbb{P}^4$.

**Proof.** The associated linear system $|2L - p|$ in $\mathbb{P}^2$ has no basepoints and for any $q$ (including the case where $q$ is infinitely near $p$), $|2L - p - q|$ has no unassigned basepoints. So $|2\pi^*L - E|$ is very ample. Note that

$$\dim|2\pi^*L - E| = \dim|2L - p|$$

is dimension of the space of conics in $\mathbb{P}^2$ passing through $p$

$$= \binom{2 + 2}{2} - 1 - 1$$

$$= 4$$

So $|2\pi^*L - E|$ gives an embedding of $Bl_p\mathbb{P}^2$ to $\mathbb{P}^4$. \(\square\)

**Proposition 2.4.** (1) The image of $Bl_p\mathbb{P}^2$ in $\mathbb{P}^4$ is of degree 3

(2) Let $X$ be a plane quintic curve with one node $p$. The strict transform $\tilde{X}$, which through the embedding of $Bl_p\mathbb{P}^2$ into $\mathbb{P}^4$ can be regarded as a curve in $\mathbb{P}^4$, has degree 8.
(3) Let $L$ be any line through $p$. Then $\tilde{X} \cdot \tilde{L} = 3$ and thus $\tilde{L}$ is a trisecant of $\tilde{X}$. In fact any trisecant of $\tilde{X}$ must be a strict transform of some line which passes through $p$.

Proof. (1) The degree is given by the number of points in the intersection of $\text{Bl}_p\mathbb{P}^2$ and two other hyperplanes in $\mathbb{P}^4$. Since $\text{Bl}_p\mathbb{P}^2$ is embedded into $\mathbb{P}^4$ by $|2\pi^*L - E|$, any hyperplane section is linearly equivalent to $2\pi^*L - E$. So

$$\text{degree} = (2\pi^*L - E)^2$$
$$= 2^2 - 1$$
$$= 3$$

(2) $\tilde{X}$ is linearly equivalent to $5\pi^*L - 2E$. The degree is the number of points in the intersection of $\tilde{X}$ and a hyperplane, which is

$$(5\pi^*L - 2E) \cdot (2\pi^*L - E) = 5 \cdot 2 - 2 \cdot 1 = 8$$

(3) Note that $\tilde{L} \sim \pi^*L - E$. So

$$\tilde{L} \cdot \tilde{X} = (\pi^*L - E) \cdot (5\pi^*L - 2E)$$
$$= 5 - 2$$
$$= 3$$

Conversely, suppose $L'$ is a trisect. Then $L' \sim \pi^*L - rE$, $r = 0,1$. $L' \cdot \tilde{X} = 3$ implies that $r = 1$. So $L'$ must be the strict transform of $L$ passing through $p$.

Proposition 2.5. Let $C$ be a curve of genus 5 and degree 8 in $\mathbb{P}^4$. If $C$ is hyperelliptic, then it lies on a hyperplane.

Proof. Let $i : C \hookrightarrow \mathbb{P}^4$ be the embedding, and $E$ be an effective divisor on $C$ such that $O_C(E) \cong i^*O_{\mathbb{P}^4}(1)$. Since $C$ is hyperelliptic, $K_C$ is not very ample. So $E \sim K_X$. By Riemann-Roch,

$$h^0(E) - h^0(K_X - E) = 8 + 1 - 5 = 4$$

Note that $\deg(K_X - E) = 8 - 8 = 0$. So $h^0(K_X - E) = 0$ and $h^0(E) = 4$, i.e. $\dim(E) = 3$. Therefore $C$ lies on a hyperplane.

Corollary 2.6. Any plane quintic curve with a single node is birational to a genus 5 nonhyperelliptic curve.
Proof. It suffices to show that $\tilde{X}$ is nonhyperelliptic. Note that $\tilde{X} \sim 5\pi^*L - 2E$, which in no way can be contained in the hyperplane section $2\pi^*L - E$ of $\text{Bl}_p\mathbb{P}^2$. The conclusion follows from Proposition 2.5.

**Example 2.7.** $p = [1 : 0 : 0]$, $X = V(x^3yz + y^5 + z^5)$.

$$\text{Bl}_p\mathbb{P}^2 = \{([x : y : z], [u : v]) \in \mathbb{P}^2 \times \mathbb{P}^1 | yv = zu\}$$

$$\varphi: \text{Bl}_p\mathbb{P}^2 \to \mathbb{P}^4$$

$$\{[x : y : z], [u : v] \mapsto [yu : zv : yv : xu : xv]\}$$

Taking $a, b, c, d, e$ as coordinates for $\mathbb{P}^4$, we have that $\text{Bl}_p\mathbb{P}^2 = V(ab - c^2, ae - cd, bd - ce)$. The equations of $\pi^*X$ are $a^5 + c^5 + acd^3 = 0$ and $b^5 + c^5 + bce^3 = 0$. To get the equation of the strict transform $\tilde{X}$, we plug in $ab - c^2 = 0$ and $ae - cd = 0$ and get $a^3 + b^2c + d^2e = 0$ and $b^3 + a^2c + de^2$. So $\tilde{X} = V(ab - c^2, ae - cd, bd - ce, a^3 + b^2c + d^2e, b^3 + a^2c + de^2)$.

**Example 2.8.** Let $X$ be a sextic plane curve with 5 nodes $p_1, \ldots, p_5$ in general position. By Theorem 4.6, Chapter V, Hartshorne, $|3\pi^*L - E_1 - \cdots - E_5|$ is very ample, and its dimension is 4. So it gives an embedding of $\text{Bl}_{p_1, \ldots, p_5}\mathbb{P}^2$ into $\mathbb{P}^4$ of degree

$$(3\pi^*L - E_1 - \cdots - E_5)^2 = 9 - 5 = 4$$

The strict transform of $X$, $\tilde{X}$, can then be embedded into $\mathbb{P}^4$. Its degree is

$$(6\pi^*L - 2E_1 - \cdots - 2E_5) \cdot (3\pi^*L - E_1 - \cdots - E_5) = 8$$

3. **Genus 5 curves in $\mathbb{P}^3$**

### 3.1. Degree 7

By Theorem 6.4, Chapter IV in Hartshorne, the minimal degree of a curve of genus 5 in $\mathbb{P}^3$ is 7.

**Proposition 3.1.** A curve of genus 5 admits an embedding of degree 7 in $\mathbb{P}^3$ iff it has no $g^1_5$.

**Proof.** Let $i: X \hookrightarrow \mathbb{P}^3$ be an embedding of degree 7 of a curve $X$ of genus 5. Let $E$ be an effective divisor such that $O_X(E) \cong i^*O_{\mathbb{P}^3}(1)$. By Riemann-Roch,

$$h^0(E) - h^0(K_X - E) = 7 + 1 - 5 = 3$$

But $h^0(E) \geq h^0(O_{\mathbb{P}^3}(1)) = 4$. Thus $h^0(K_X - E) \geq 1$ and $E$ is special. By Clifford’s theorem, $\dim |E| \leq \frac{7}{2}$. So $\dim |E| = 3$ and $\deg(K_X - E) = 1$. It follows that there exists
$P \in |K_X - E|$, i.e. $E = K - P$. Since $E$ is very ample, for any $Q$ and $R$, $1 = \dim |E| - 2 = \dim |K - P| - 2 = \dim |K - P - Q - R|$. But by Riemann-Roch again,

$$\dim |P + Q + R| - \dim |K - P - Q - R| = 3 + 1 - 5 = -1$$

So $\dim |P + Q + R| = 0$ for all $Q$ and $R$. This is only possible if $X$ does not have a $g^1_3$. □

**Corollary 3.2.** Curves of genus 5 and degree 7 in $\mathbb{P}^3$ are not birational to plane quintic curves with one node.

**Proposition 3.3.** For any curve of genus 5 and degree 7 in $\mathbb{P}^3$, there exists a nonsingular cubic surface which contains it.

**Proof.** Consider the short exact sequence of sheaves

$$0 \to I_X \to O_{\mathbb{P}^3} \to O_X \to 0$$

Twisting by 3 and taking the long exact sequence of sheaf cohomology, we have

$$0 \to H^0(I_X(3)) \to H^0(O_{\mathbb{P}^3}(3)) \to H^0(O_X(3)) \to \cdots$$

Note that $h^0(O_{\mathbb{P}^3}(3)) = \binom{3 + 3}{3} = 20$. Using Riemann-Roch,

$$h^0(O_X(3)) - h^0(\omega_X \otimes O_X(-3)) = 3 \cdot 7 + 1 - 5 = 17$$

Since $\text{deg}(\omega_X \otimes O_X(-3)) \leq 0$, $h^0(\omega_X \otimes O_X(-3)) = 0$ and $h^0(O_X(3)) = 17$. So $h^0(I_X(3)) \geq 3$ and the space of cubic forms vanishing on $X$ is at least 2. One can choose a cubic form whose vanishing set is a smooth cubic surface containing $X$. □

**Proposition 3.4.** Let $X$ be a genus 5 curve of degree 7 on a nonsingular cubic surface $S$ in $\mathbb{P}^3$. Regarding $S$ as $\mathbb{P}^2$ with 6 points $p_1, \ldots, p_6$ in general position blown up, $X$ is linearly equivalent to one of the following 3 divisors

1. $6\pi^*L - 2 \sum_{i=1}^{5} E_i - E_6$. In this case $X$ is the strict transform of a plane sextic curve with 5 nodes $p_1, \ldots, p_5$ and passing through $p_6$.
2. $7\pi^*L - 3E_1 - 3E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6$
3. $8\pi^*L - 3 \sum_{i=1}^{5} E_i - 2E_6$
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Proof. Let $X \sim a\pi^*L - \sum_{i=1}^6 b_i E_i$. Then

$$7 = \deg X$$

$$= (a\pi^*L - \sum_{i=1}^6 b_i E_i) \cdot K_S$$

$$= (a\pi^*L - \sum_{i=1}^6 b_i E_i) \cdot (-3\pi^*L + \sum_{i=1}^6 E_i)$$

(1) $\implies 3a - \sum_{i=1}^6 b_i = 7$

$$5 = p_g(X)$$

$$= 1 + \frac{X \cdot (X + K_X)}{2}$$

$$= 1 + \frac{(a\pi^*L - \sum_{i=1}^6 b_i E_i) \cdot ((a - 3)\pi^*L - \sum_{i=1}^6 (b_i - 1)E_i)}{2}$$

$$= 1 + \frac{a(a - 3) - \sum_{i=1}^6 (b_i^2 - b_i)}{2}$$

(2) $\implies a^2 - 3a - \sum_{i=1}^6 b_i^2 + \sum_{i=1}^6 b_i = 8$

Plugging in (1) to (2),

$$a^2 - 3a - \sum_{i=1}^6 b_i^2 + 3a - 7 = 8$$

$$a^2 - \sum_{i=1}^6 b_i^2 = 15$$

By Schwarz’s inequality,

$$6 \sum_{i=1}^6 b_i^2 \geq \left( \sum_{i=1}^6 b_i \right)^2$$

$$6(a^2 - 15) \geq (3a - 7)^2$$

$$3a^2 - 42a + 139 \leq 0$$

Since $a$ is an integer, it can only be 6, 7 or 8. By trial and error, we get $b_i$ for each of the three possible cases for $a$. □
Example 3.5. Let $\tilde{X}$ be the strict transform of a plane sextic curve with 5 nodes $p_1, \ldots, p_5$ and passing through $p_6$. Then $\deg(\tilde{X}) = 7$ and $p_g(\tilde{X}) = 5$. Let $i : \tilde{X} \to S$ be the embedding. By Adjunction Formula,

$$\omega_{\tilde{X}} = i^*(K_S \otimes \mathcal{O}_S(\tilde{X}))$$

$$= i^*(\mathcal{O}_S(-3\pi^* L + E_1 + \cdots + E_6 + 6\pi^* L - 2E_1 - \cdots - 2E_5 - E_6))$$

$$= i^*(\mathcal{O}_S(3\pi^* L - E_1 - \cdots - E_5))$$

By Theorem 4.6, Chapter V, Hartshorne, $3\pi^* L - E_1 - \cdots - E_5$ is very ample. So $K_{\tilde{X}}$ is very ample as well. Thus $\tilde{X}$ must be nonhyperelliptic, and

Proposition 3.6. Any plane sextic curve with 5 nodes (in general position) is birational to a nonhyperelliptic curve.

3.2. Degree 8.

Example 3.7. Consider the curve in Example 2.7. If we project the curve from $[0 : 0 : 1 : 0 : 0]$, we get a curve in $\mathbb{P}^3$ whose vanishing ideal is $(ae^2 - bd^2, a^3d + b^2ae + d^3e, a^3e + b^3d + d^2e^2, b^3e + a^2bd + de^3)$. The degree of the curve is still 8 because $[0 : 0 : 1 : 0 : 0]$ is not on the curve.

Example 3.8. Let $X$ be a plane quintic curve with one node $p_1$, and $\tilde{X}$ be the strict transform of $X$ in $S = \text{Bl}_{p_1, \ldots, p_6} \mathbb{P}^2$ where $p_2, \ldots, p_6$ are 5 nonsingular points on $X$. Then

$$\tilde{X} \sim 5\pi^* L - 2E_1 - \sum_{i=2}^{6} E_i$$

and

$$\deg(\tilde{X}) = \tilde{X} \cdot (-K_S)$$

$$= (5\pi^* L - 2E_1 - \sum_{i=2}^{6} E_i) \cdot (3\pi^* L - \sum_{i=1}^{6} E_i)$$

$$= 8$$

$$\omega_{\tilde{X}} = i^*(K_S \otimes \mathcal{O}_S \mathcal{O}(\tilde{X}))$$

$$= i^*(\mathcal{O}_S(-3\pi^* L + E_1 + \cdots + E_6 + 5\pi^* L - 2E_1 - E_2 - \cdots - E_6))$$

$$= i^*(\mathcal{O}_S(2\pi^* L - E_1))$$

As $|2\pi^* L - E_1|$ is very ample and $i$ is an embedding, $K_{\tilde{X}}$ is very ample and $\tilde{X}$ is nonhyperelliptic.
Corollary 3.9.  
(1) Any plane quintic curve with one node is birational to a nonhyperelliptic curve.
(2) Any hyperelliptic curve of genus 5 has no $g_1^3$.

Example 3.10. Let $X$ be a curve of genus 5 on a nonsingular quadric surface in $\mathbb{P}^3$. Assume it is of type $(a,b)$. Then $(a-1)(b-1) = 5$. So $(a,b) = (2,6)$ or $(6,2)$, and $X$ must be of type $(2,6)$. Its degree is $2+6 = 8$. $X$ must be hyperelliptic because projecting the quadric surface onto one of its rulings gives a 2-to-1 morphism from $X$ to $\mathbb{P}^1$. An example of $X$:

$$X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$$

$$( [s : t], [u : v]) \mapsto [su : tu : sv : tv]$$

Suppose $X$ is given by the equation $s^2v^6 - t^2(u^6 + v^6 + u^3v^3 + u^2v^4 + u^4v^2) = 0$. Using $x, y, z, w$ as coordinates of $\mathbb{P}^3$, we have that the vanishing ideal of $X$ is $(x^2z^6 - y^2(x^6 + x^4z^2 + x^3z^3 + x^2z^4 + z^6), x^2w^6 - y^2(y^6 + y^4w^2 + y^3w^3 + y^2w^4 + w^6), z^8 - w^2(x^6 + x^4z^2 + x^3z^3 + x^2z^4 + z^6), z^2w^4 - y^6 - y^4w^2 - y^3w^3 - y^2w^4 - w^6, xw - yz)$. 
