A new combinatorial Rogers-Ramanujan proof

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Outline

- What are the Rogers-Ramanujan identities?
- First step: generalized Schur's identity
- Background: Dyson's proof of Euler's pentagonal number theorem
- Basic tools: selection and insertion
- New proof of generalized Schur's identity:
  - Definition of (k,m)-rank
  - Two bijections
- Further questions

Rogers-Ramanujan identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{i=0}^{\infty} \frac{1}{(1-q^{5i+1})(1-q^{5i+4})}$$

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Combinatorial interpretation.

RHS: partitions into parts congruent to 1, 4 (mod 5)

LHS: partitions with no parts below the Durfee square

Andrews' generalization

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} = \prod_{n\not\equiv 0,\pm k \pmod{2k+1}} \frac{1}{1-q^n}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$.

- notation: $(q)_n = \prod_{j=1}^{n}(1-q^j)$

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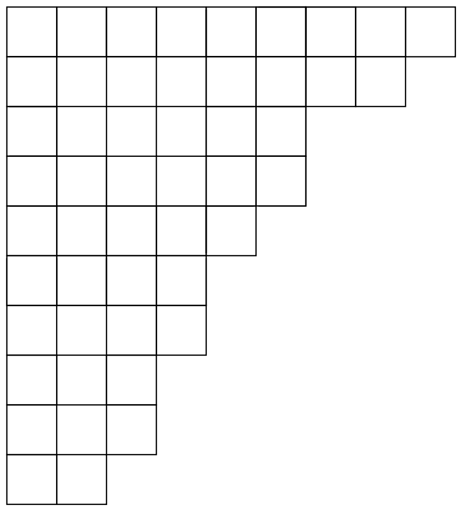
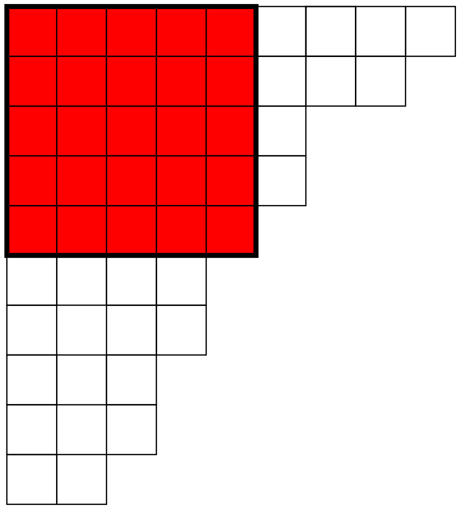
Combinatorial interpretation.

LHS: successive Durfee squares

Successive Durfee squares

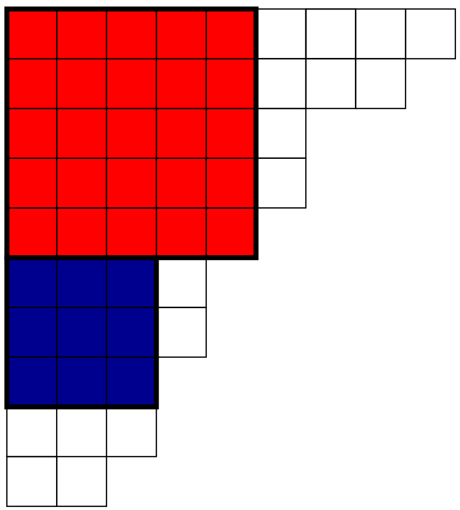


Successive Durfee squares



- first Durfee square: largest square that fits in the diagram of λ

Successive Durfee squares



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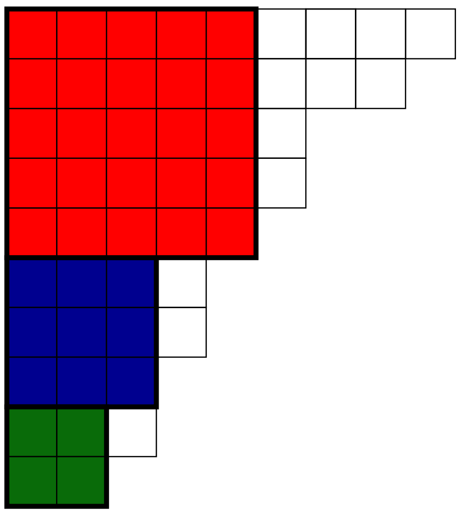

- first Durfee square: largest square that fits in the diagram of λ
- second Durfee square: largest square that fits below the first Durfee square of λ
- k th Durfee square: largest square that fits below the $(k-1)$ st Durfee square of λ

Successive Durfee squares

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} = \prod_{n \not\equiv 0, \pm k \pmod{2k+1}} \frac{1}{1-q^n}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$.

Combinatorial interpretation.

LHS: partitions with at most $(k-1)$ successive Durfee squares

First Step

- apply Jacobi triple product identity to RHS.

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$$\sum_{j=-\infty}^{\infty} z^j q^{\frac{j(j+1)}{2}} = \prod_{i=1}^{\infty}(1+zq^i)\prod_{j=0}^{\infty}(1+z^{-1}q^j)\prod_{i=1}^{\infty}(1-q^i)$$

specializes to

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(j+1)(2k+1)}{2}-kj} = \prod_{n\equiv 0,\pm k \pmod{2k+1}} (1-q^n)$$

First Step

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this changes the RHS of Rogers-Ramanujan to give

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} =$$

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- equivalent to the generalized Rogers-Ramanujan identity
- we will call this generalized Schur's identity

Background: Dyson's proof

- Euler's pentagonal number theorem

$$1 = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(3j+1)}{2}}$$

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- this is the case $k = 1$ of our generalized Schur's identity
- we will mimic Dyson's proof of Euler's pentagonal number theorem

Background: Dyson's proof

Def. rank(λ) = largest part - number of parts

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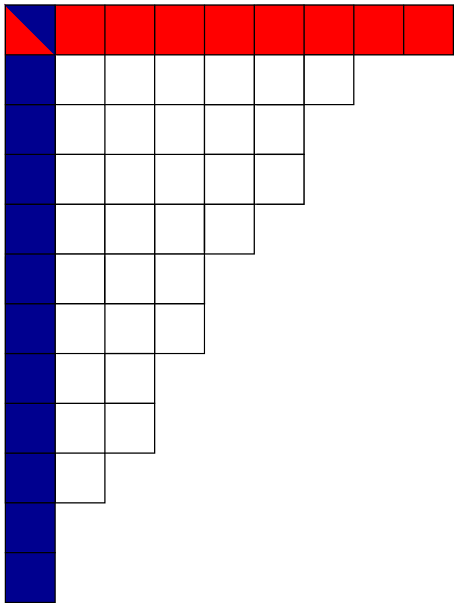

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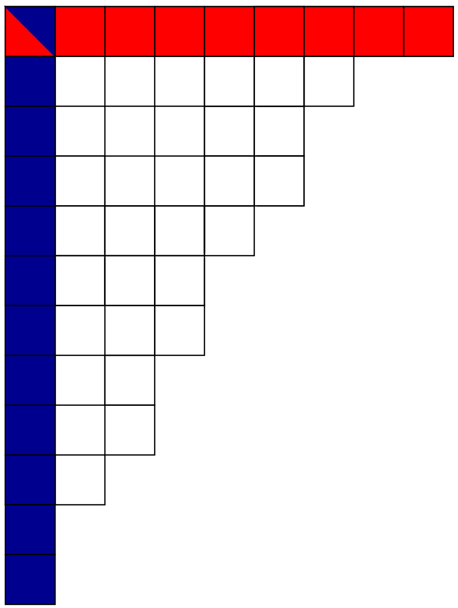

- rank$(\lambda) = 9 - 12 = -3$

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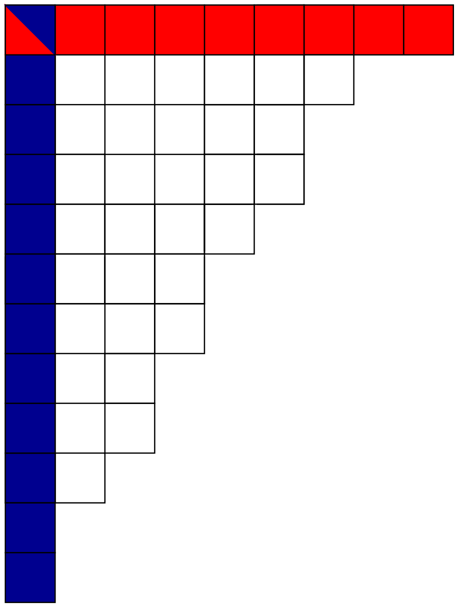

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Conjugation

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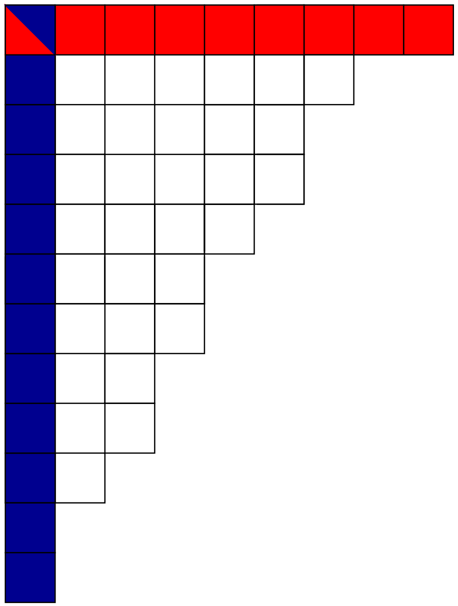

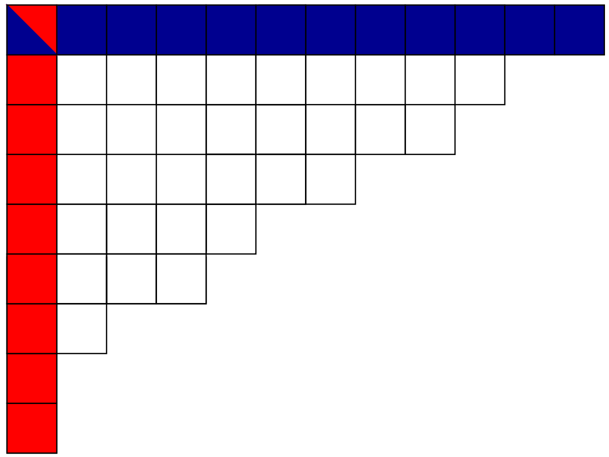

- rank$(\lambda) = 9 - 12 = -3$
- $\lambda \vdash n$

- rank$(\lambda') = 12 - 9 = 3$
- $\lambda' \vdash n$

Conjugation

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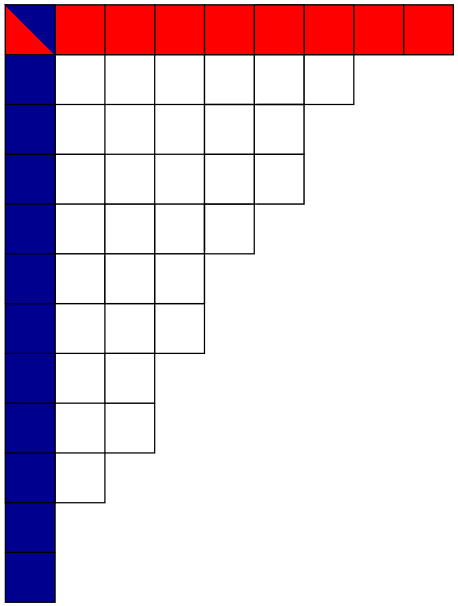

- rank$(\lambda) = -3 \leq -r$
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Dyson's map: ϕ_r

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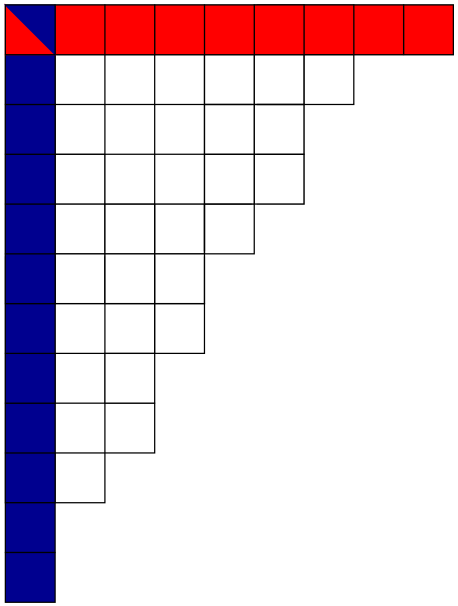

- rank$(\lambda) = -3 \leq -r$
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- rank$(\phi_r(\lambda)) \geq -r-2$
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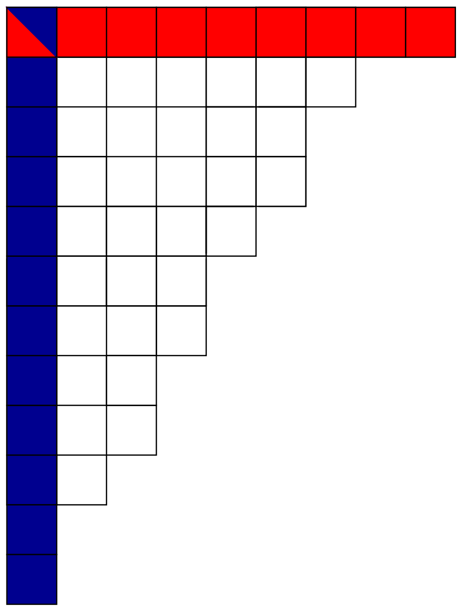

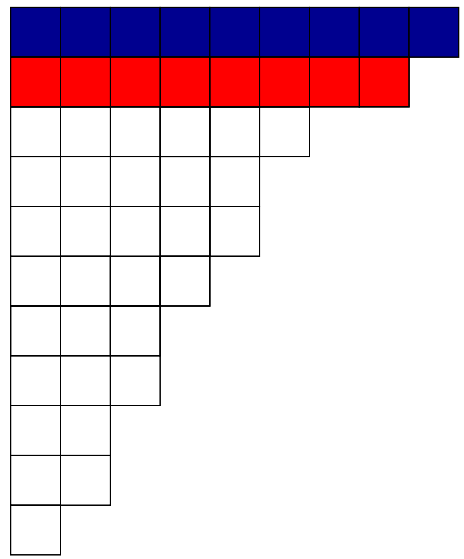

- rank$(\lambda) = -3 \leq -r$
- $\lambda \vdash n$

- rank$(\phi_r(\lambda)) \geq -r - 2$
- $\phi_r(\lambda) \vdash n - r - 1$

Dyson's map: ϕ_r (eg. $r = 2$)

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$$h(n, r) = h(n, -r)$$

- Dyson's map

$$h(n, \leq -r) = h(n - r - 1, \geq -r - 2)$$

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- conjugation

  $$h(n,r) = h(n,-r)$$

- Dyson's map

  $$h(n, \leq -r) = h(n-r-1, \geq -r-2)$$

- one more observation ...

  $$h(n, \leq -r) + h(n, \geq -r+1) = p(n)$$

  where $p(n)$ = the number of partitions of n

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$$h(n,\leq -r) + h(n,\geq -r+1) = p(n)$$

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Simple algebra gives the generating function for partitions with rank at most $-r$ and proves Euler's pentagonal number theorem

Plan

To prove the generalized Schur's identity:

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} =$$

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)} \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(j+1)(2k+1)}{2}-kj}$$

Plan

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- give a new definition of rank
- find map which generalizes conjugation
- find map which generalizes Dyson's map
- simple algebra will complete the proof

Selection

Defn. $f(\lambda)$ = first part of λ

Given a sequence of k partitions, $\lambda^1, \lambda^2, ..., \lambda^k$,

and $k-1$ nonnegative integers, $p_2, p_3, ..., p_k$,

such that $f(\lambda^2) \leq p_2, f(\lambda^3) \leq p_3, ..., f(\lambda^k) \leq p_k$.

Select one row from each partition as follows:

- Select the first (that is, the largest) part of λ^k.
- Suppose we have selected the jth part of λ^i, then select the $(j + p_i - {\lambda^i}_j)$th part of λ^{i-1}.

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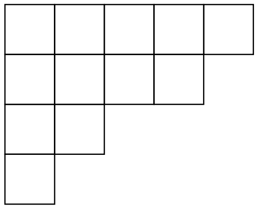
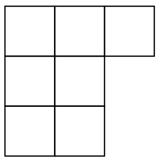
Defn. $A(\lambda^1, \lambda^2, ..., \lambda^k; p_2, p_3, ..., p_k)$ = sum of the selected parts

Selection

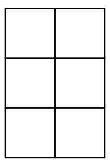
λ^1



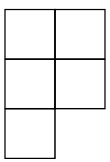
λ^2



λ^3

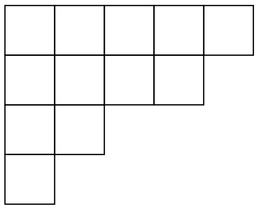


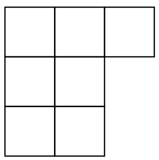
λ^4

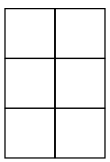


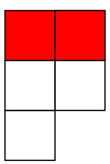
$A(\lambda^1, \lambda^2, \lambda^3, \lambda^4; 4, 2, 3)$

Selection

λ^1 

λ^2 

λ^3 

λ^4 

$A(\lambda^1, \lambda^2, \lambda^3, \lambda^4; 4, 2, 3)$

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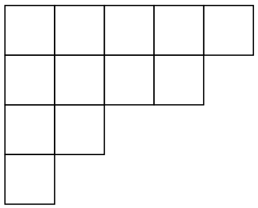

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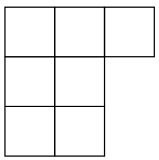

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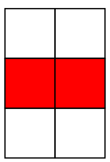

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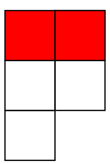

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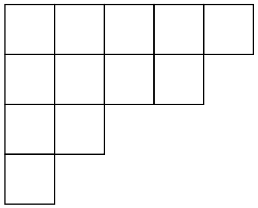

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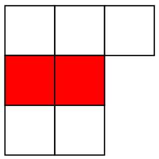

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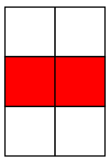

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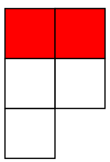

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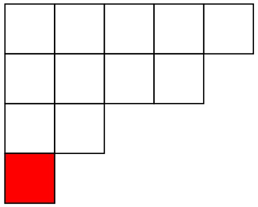

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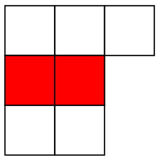

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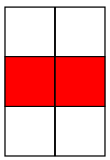

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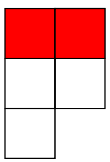

$A(\lambda^1, \lambda^2, \lambda^3, \lambda^4; 4, 2, 3)$

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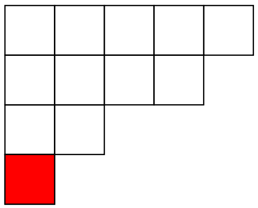

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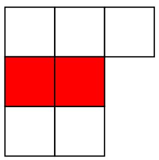

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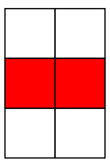

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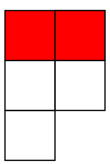
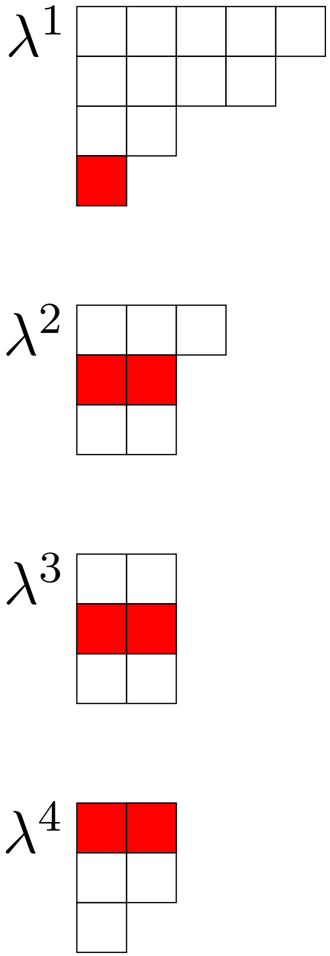


$$A(\lambda^1, \lambda^2, \lambda^3, \lambda^4; 4, 2, 3) = 7$$

Selection



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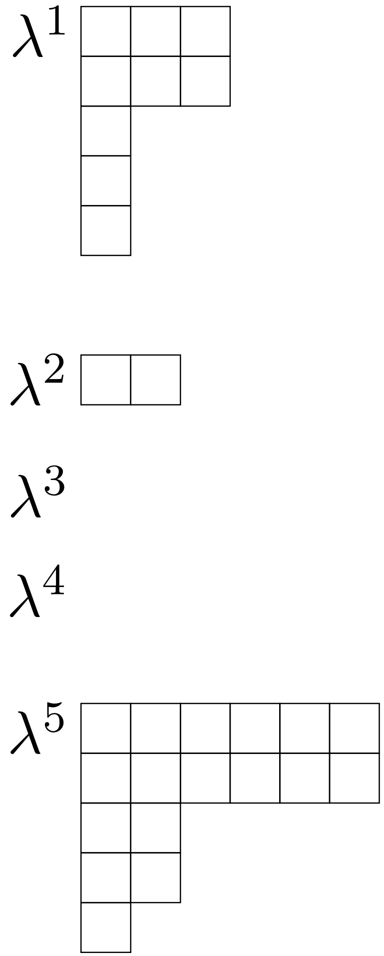
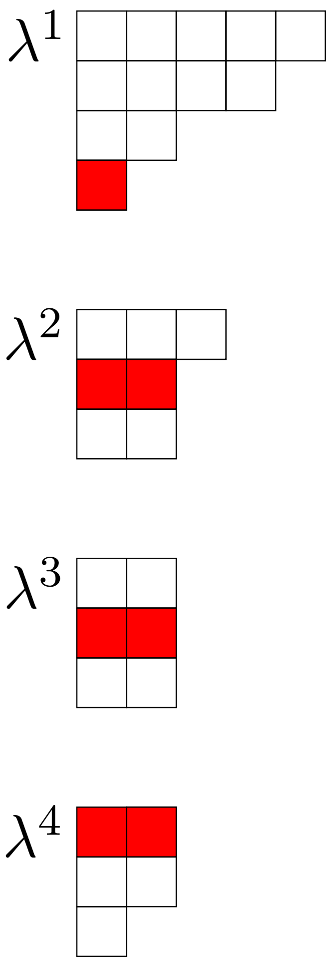
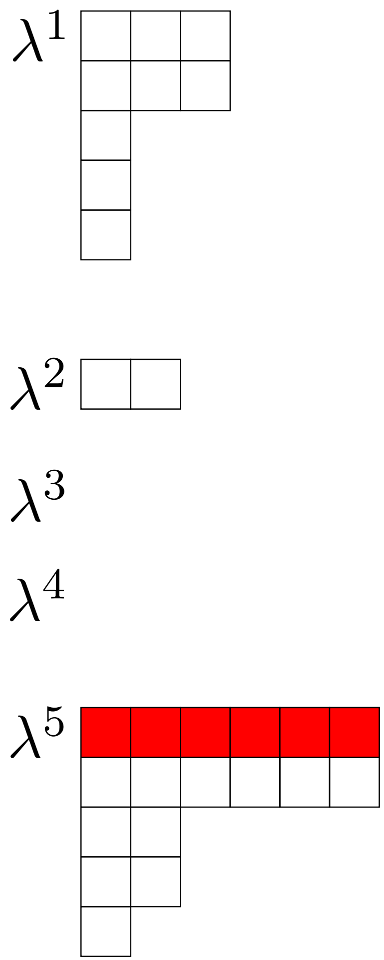


$A(\lambda^1, ..., \lambda^5; 2, 0, 2, 6)$

Selection

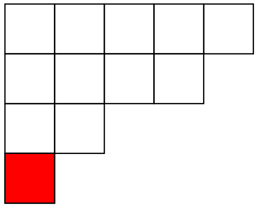


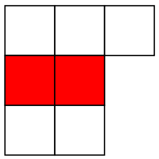
$$A(\lambda^1, \lambda^2, \lambda^3, \lambda^4; 4, 2, 3) = 7$$

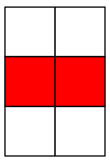


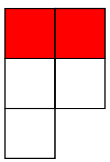
$$A(\lambda^1, ..., \lambda^5; 2, 0, 2, 6)$$

Selection

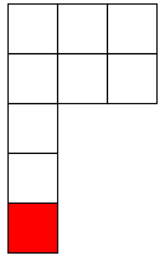
λ^1 

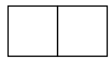
λ^2 

λ^3 

λ^4 

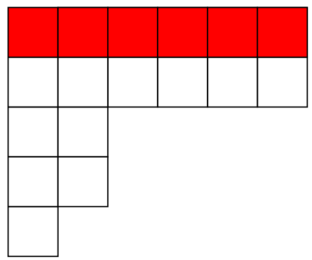
$$A(\lambda^1, \lambda^2, \lambda^3, \lambda^4; 4, 2, 3) = 7$$

λ^1 

λ^2 

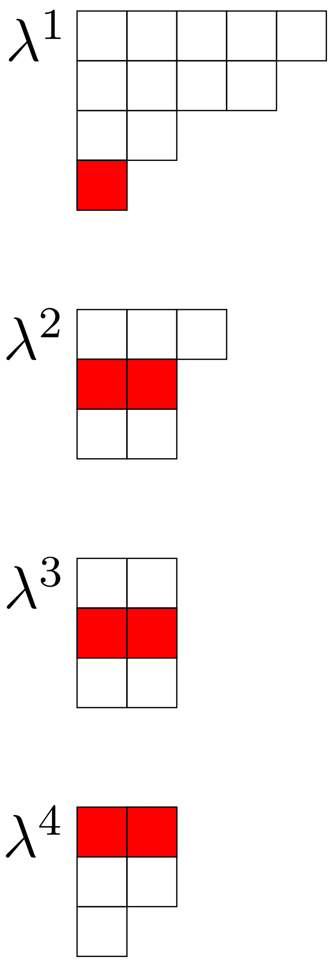
λ^3

λ^4

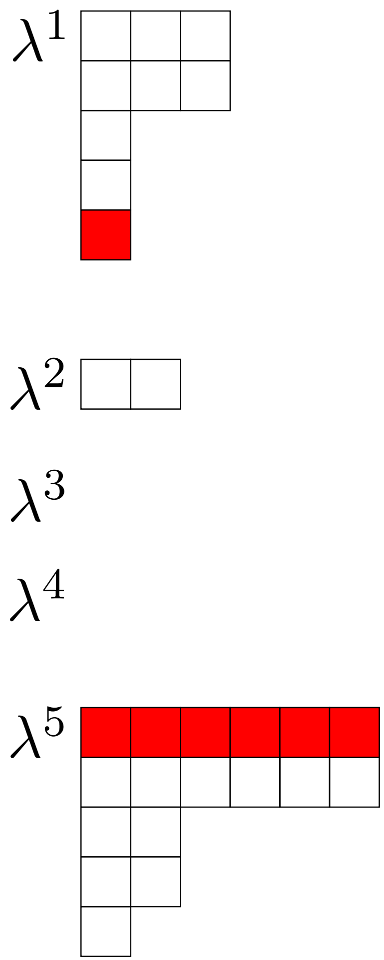
λ^5 

$$A(\lambda^1, ..., \lambda^5; 2, 0, 2, 6)$$

Selection



$$A(\lambda^1, \lambda^2, \lambda^3, \lambda^4; 4, 2, 3) = 7$$



$$A(\lambda^1, ..., \lambda^5; 2, 0, 2, 6) = 7$$

Insertion

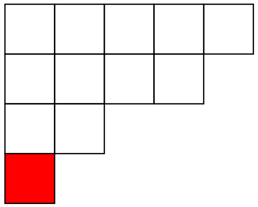
Given a sequence of k partitions, $\lambda^1, \lambda^2, ..., \lambda^k$,
$k-1$ nonnegative integers, $p_2, p_3, ..., p_k$,
such that $f(\lambda^2) \leq p_2, f(\lambda^3) \leq p_3, ..., f(\lambda^k) \leq p_k$, and
an integer $a \geq A(\lambda^1, \lambda^2, ..., \lambda^k; p_2, p_3, ..., p_k)$,

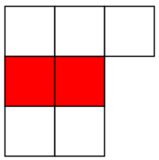
there exists a unique sequence of k partitions, $\mu^1, \mu^2, ..., \mu^k$, obtained by inserting one (possibly empty) part to each of the original partitions, $\lambda^1, \lambda^2, ..., \lambda^k$, such that

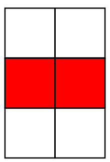
- $|\mu^1| + |\mu^2| + ... + |\mu^k| = n + a$,
- $f(\mu^2) \leq p_2, f(\mu^3) \leq p_3, ..., f(\mu^k) \leq p_k$, and
- $A(\mu^1, \mu^2, ..., \mu^k; p_2, p_3, ..., p_k) = a$.

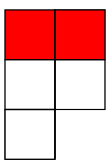
Moreover, the inserted parts are those which are selected when calculating $A(\mu^1, \mu^2, ..., \mu^k; p_2, p_3, ..., p_k)$.

Insertion

λ^1 

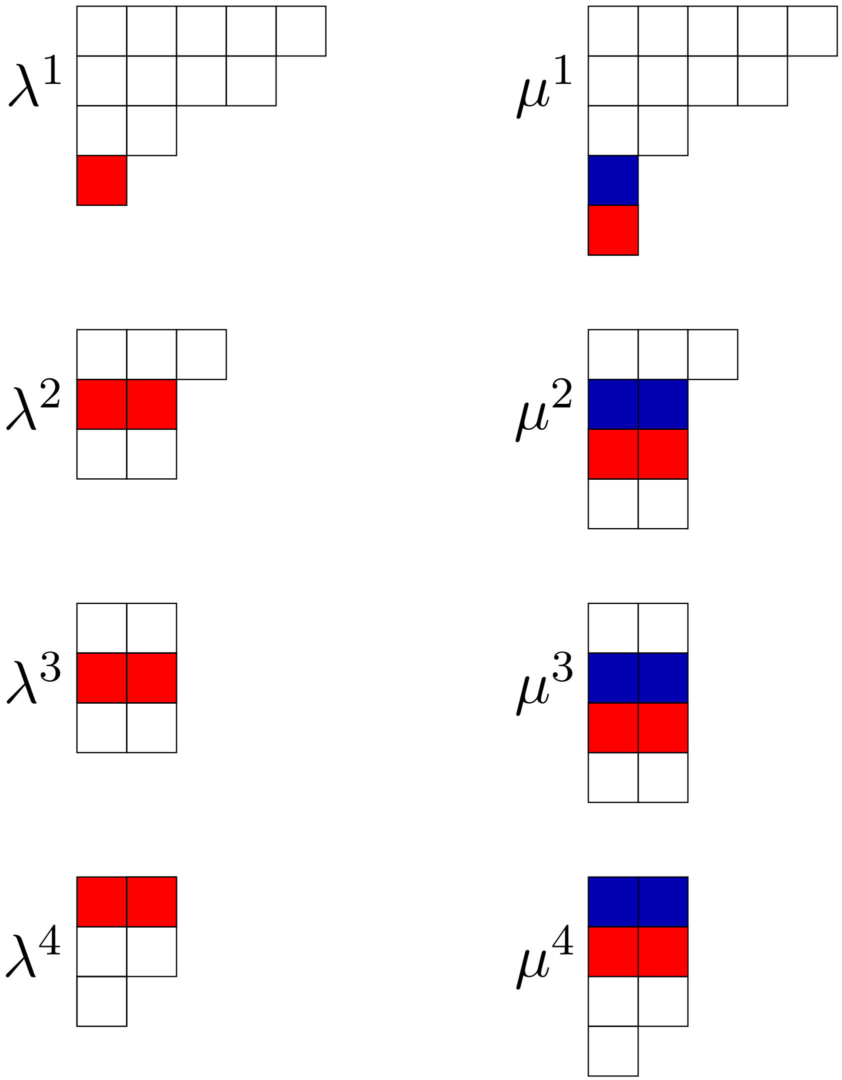
λ^2 

λ^3 

λ^4 

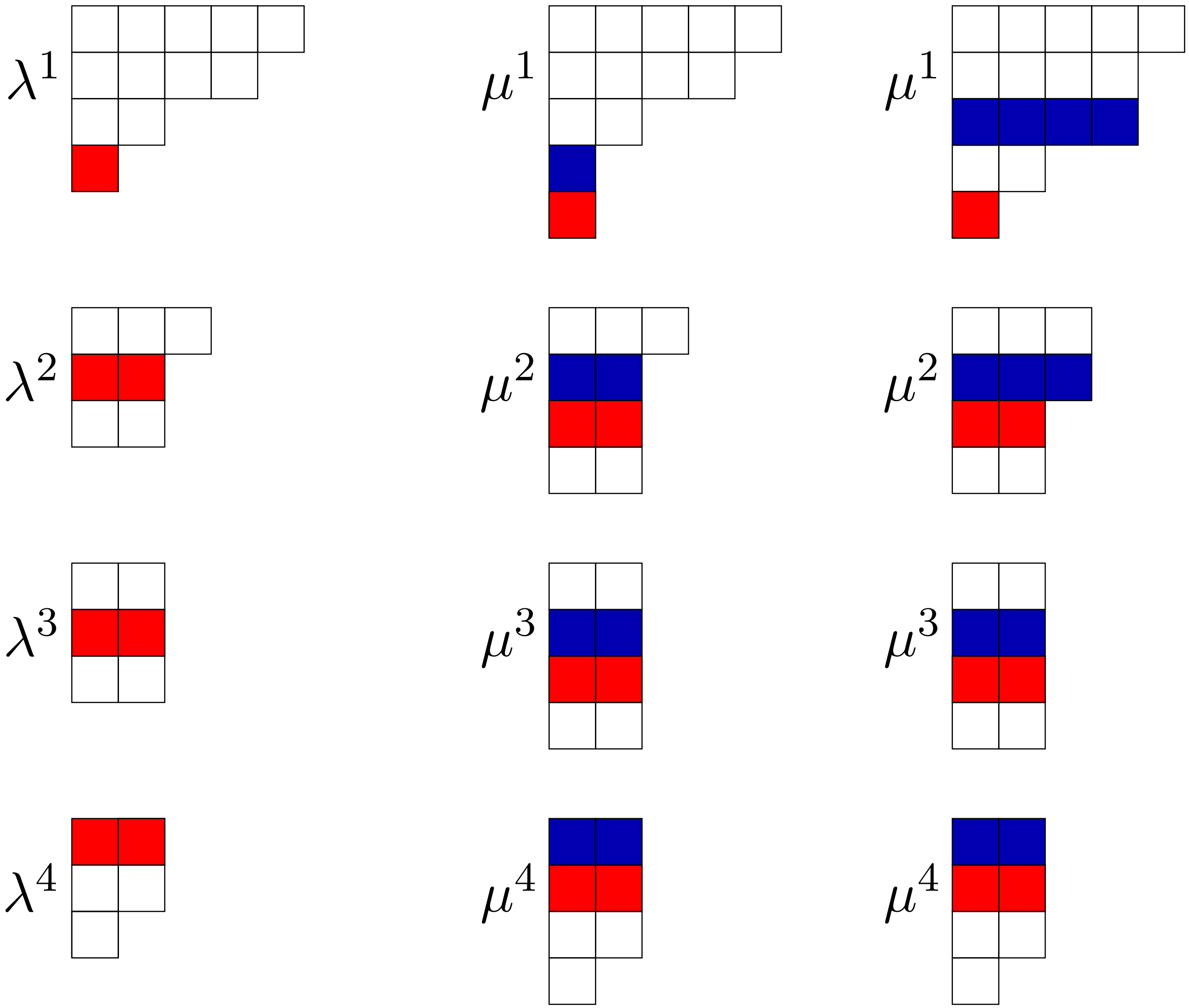
$A(\lambda; 4, 2, 3) = 7$

Insertion



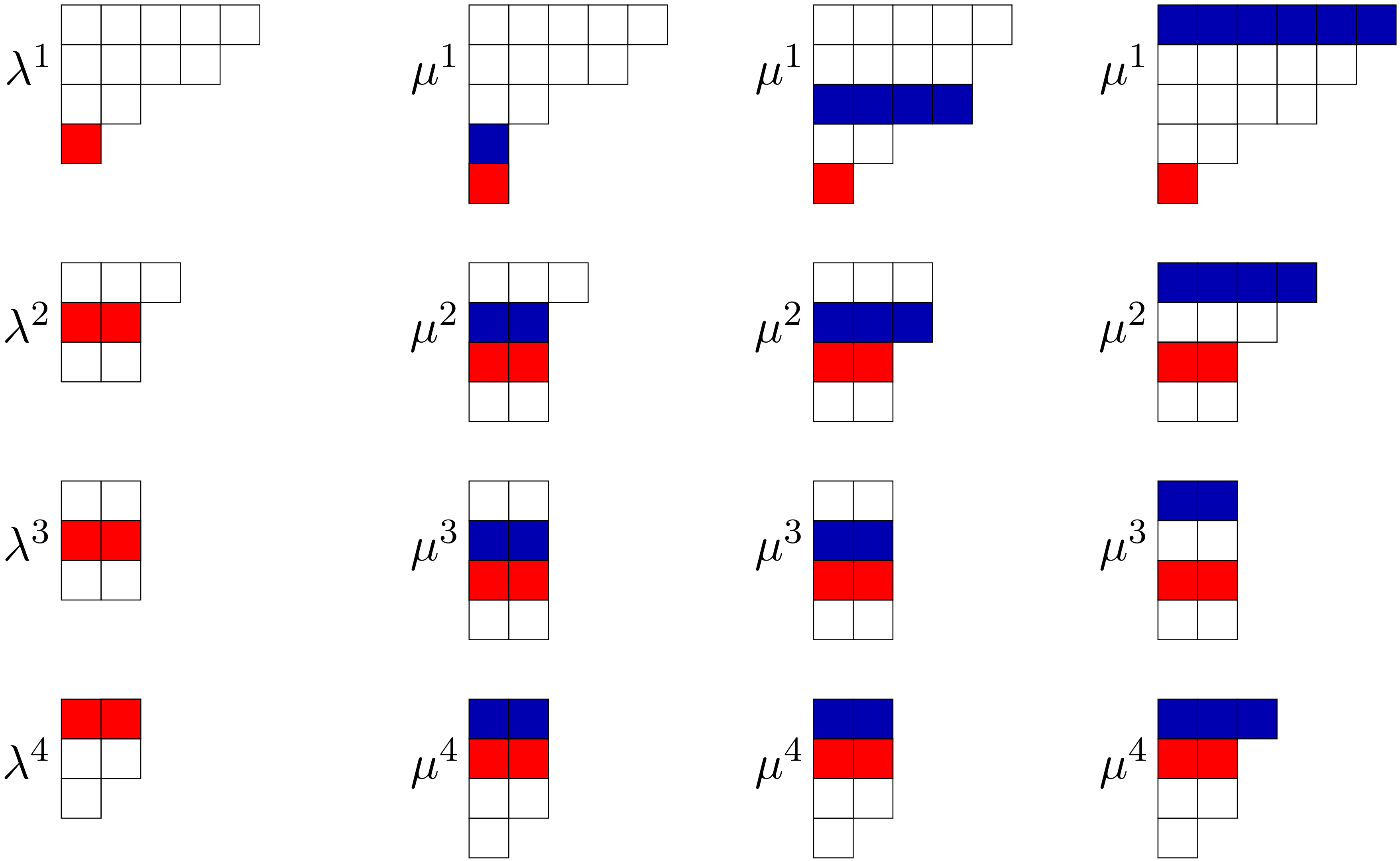
$A(\lambda; 4, 2, 3) = 7 \qquad a = 7$

Insertion



$A(\lambda; 4, 2, 3) = 7$ $\quad a = 7$ $\quad a = 11$

Insertion



$A(\lambda; 4, 2, 3) = 7$ $\quad a = 7$ $\quad a = 11$ $\quad a = 16$

Successive m-Durfee rectangles

Def. m-rectangle: a rectangle with height = width + m

Successive m-Durfee rectangles

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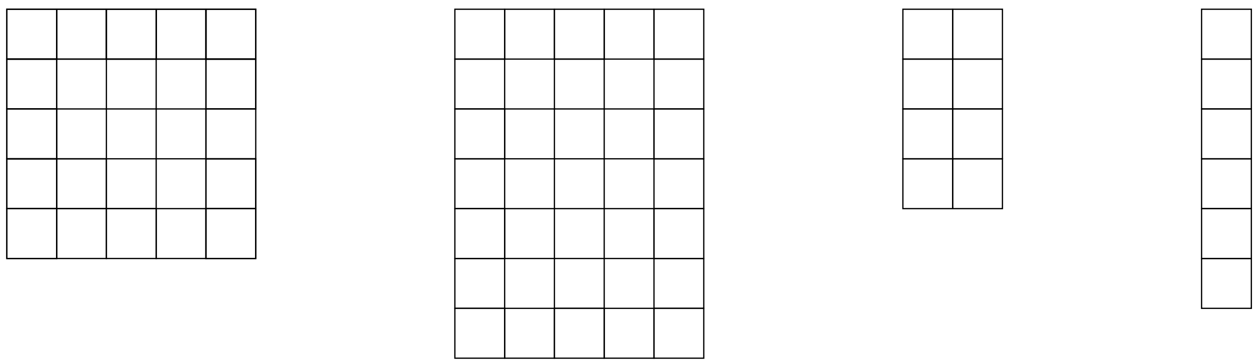

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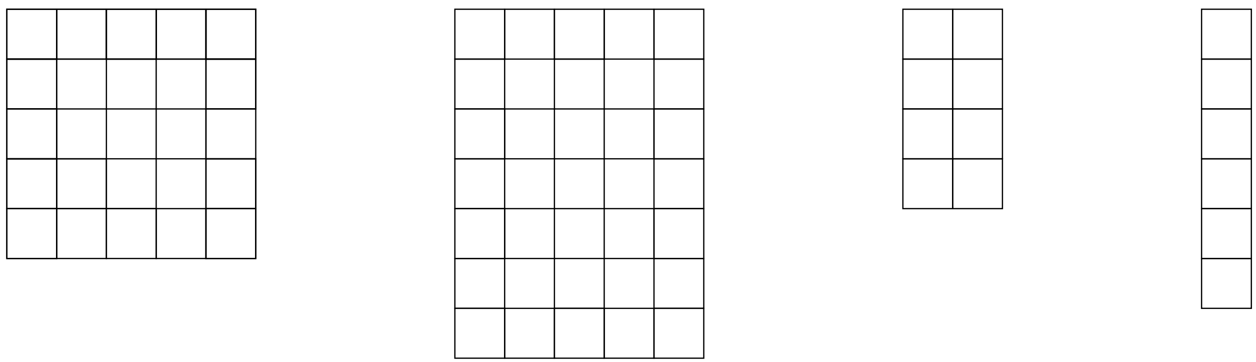
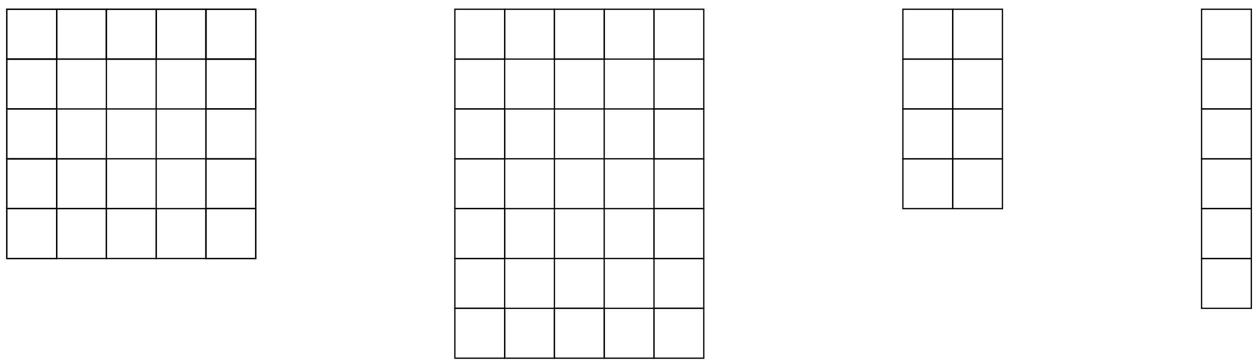


$m = 0$

Successive m-Durfee rectangles

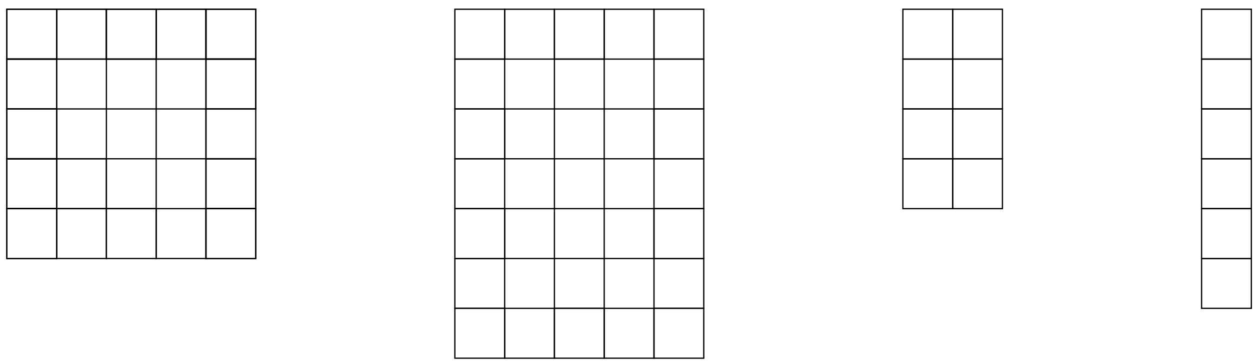
Def. m-rectangle: a rectangle with height = width + m



$m = 0$ $\qquad$ $m = 2$

Successive m-Durfee rectangles

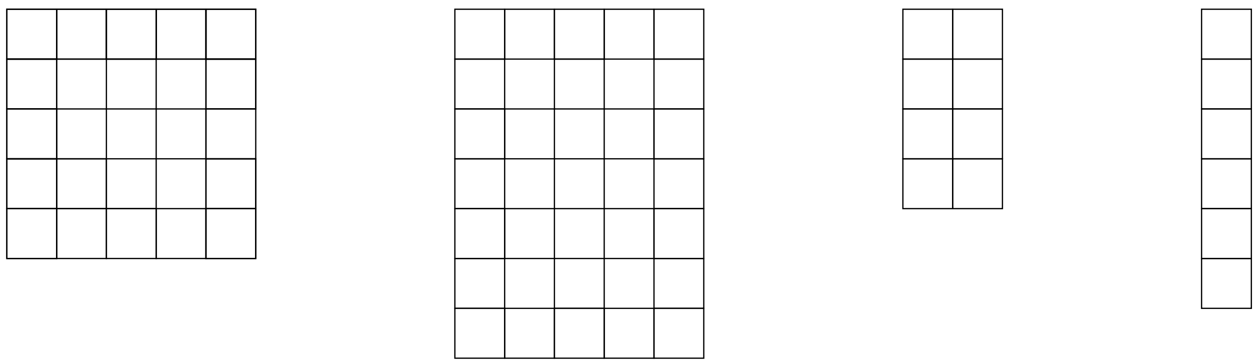
Def. m-rectangle: a rectangle with height = width + m



$m = 0$ $\qquad$ $m = 2$ $\qquad$ $m = 2$

Successive m-Durfee rectangles

Def. m-rectangle: a rectangle with height = width + m



$m = 0$ $m = 2$ $m = 2$ $m = 5$

Successive m-Durfee rectangles

Def. successive m-Durfee rectangle of λ:

Successive m-Durfee rectangles

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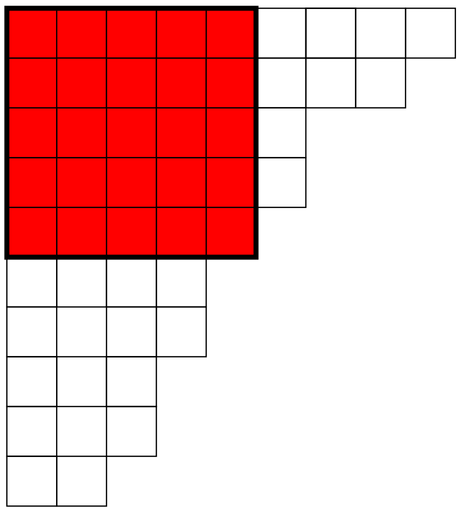
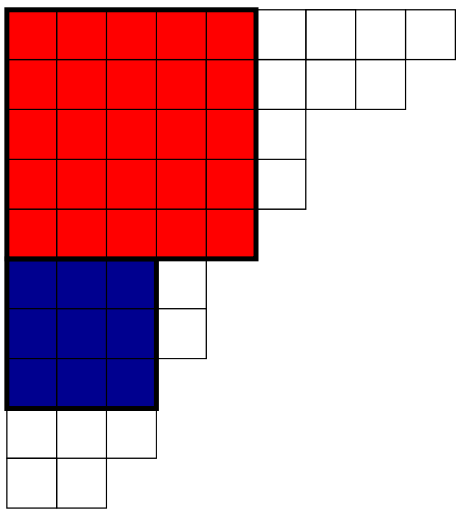


$m = 0$

Successive m-Durfee rectangles

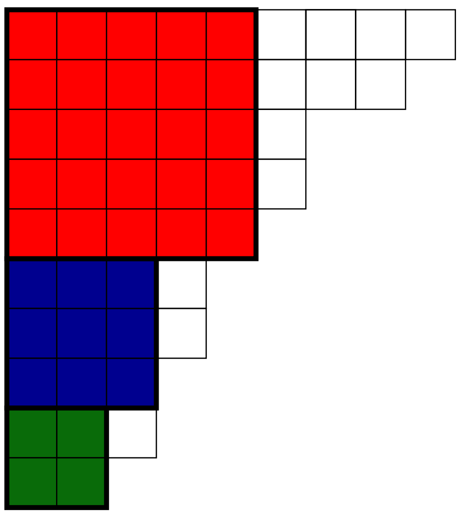
Def. successive m-Durfee rectangle of λ:



$m = 0$

Successive m-Durfee rectangles

Def. successive m-Durfee rectangle of λ:



$m = 0$

Successive m-Durfee rectangles

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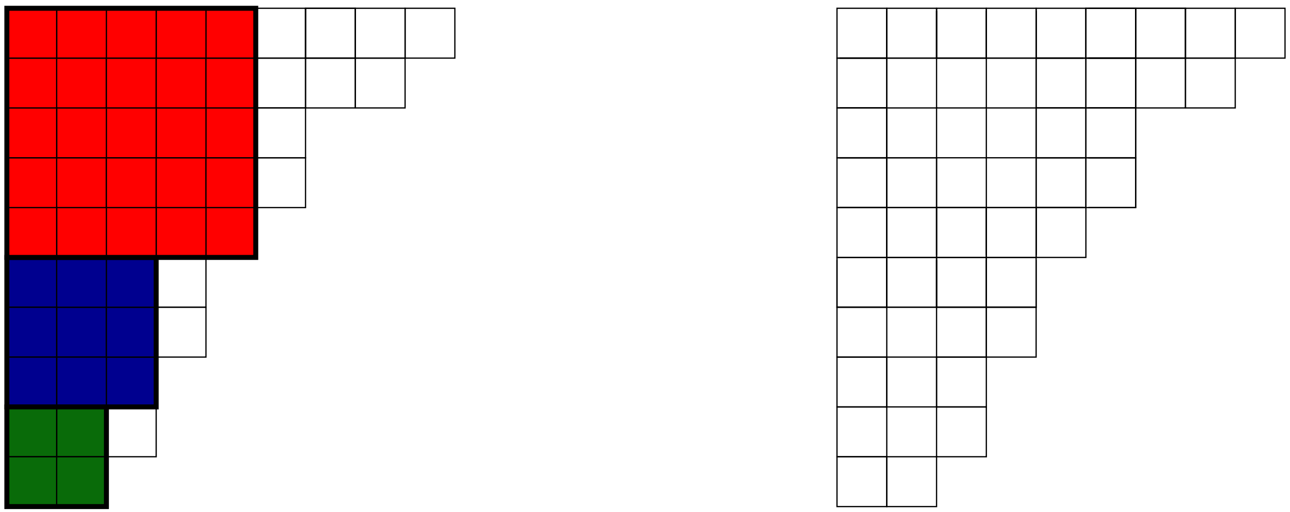
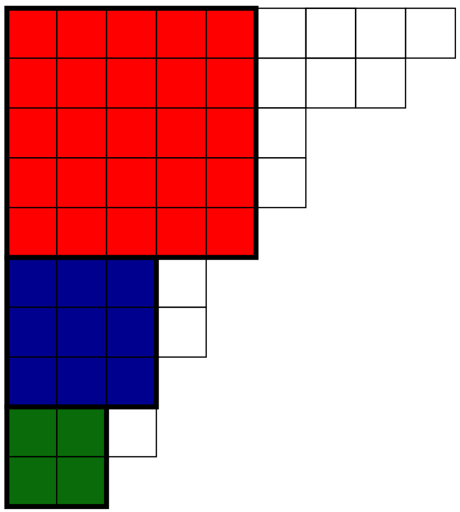


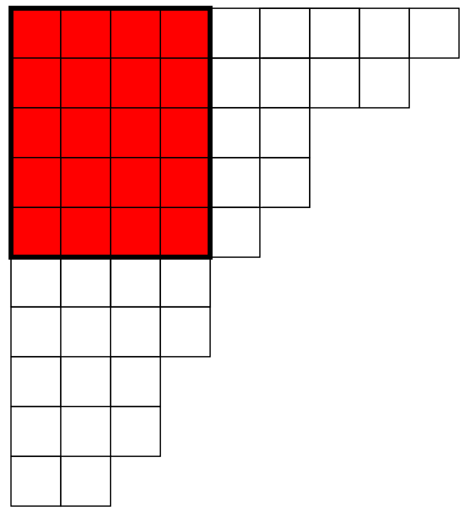
$m = 0$ $\qquad\qquad$ $m = 1$

Successive m-Durfee rectangles

Def. successive m-Durfee rectangle of λ:



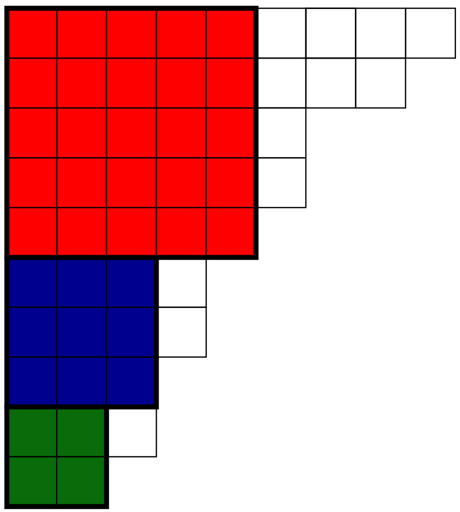
$m = 0$



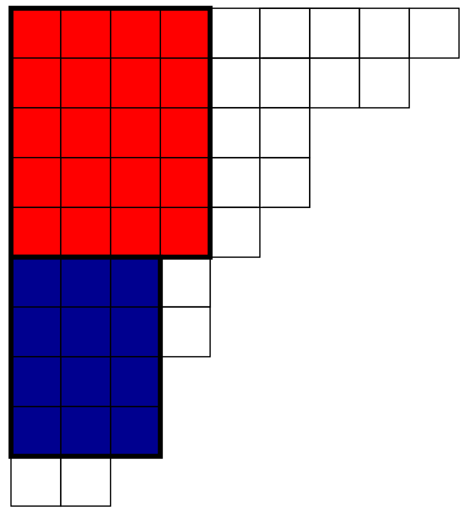
$m = 1$

Successive m-Durfee rectangles

Def. successive m-Durfee rectangle of λ:



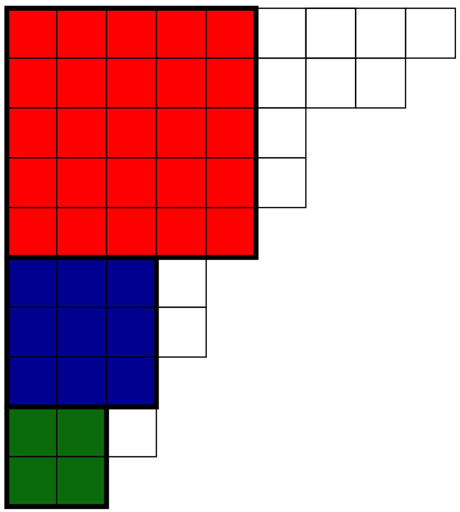
$m = 0$



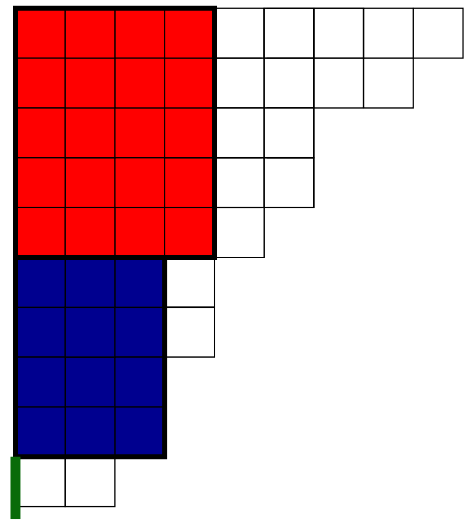
$m = 1$

Successive m-Durfee rectangles

Def. successive m-Durfee rectangle of λ:



$m = 0$



$m = 1$

Dyson's rank

Defn. rank(λ) = largest part - number of parts = $f(\lambda) - \ell(\lambda)$

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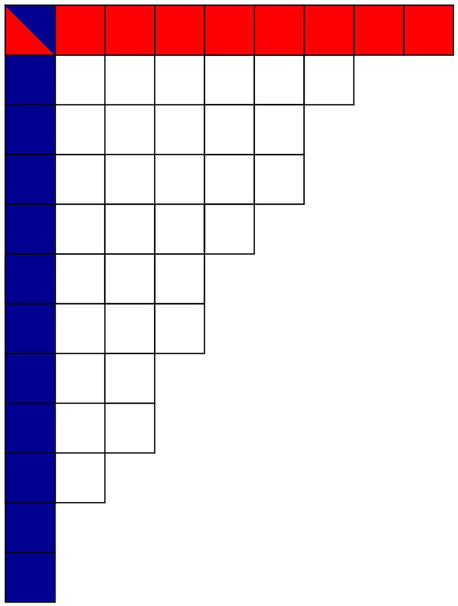

Dyson's rank

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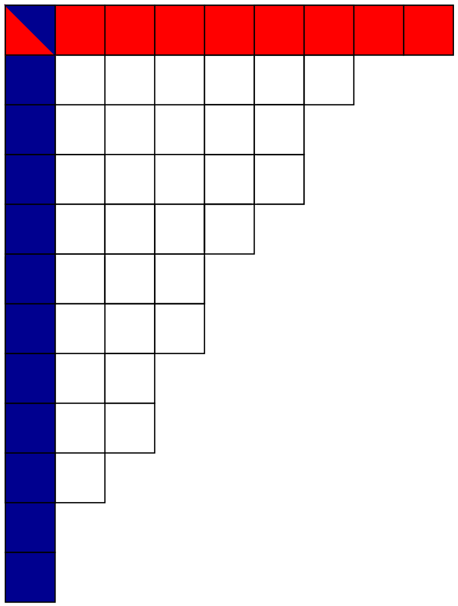

- rank$(\lambda) = 9 - 12 = -3$

Dyson's rank

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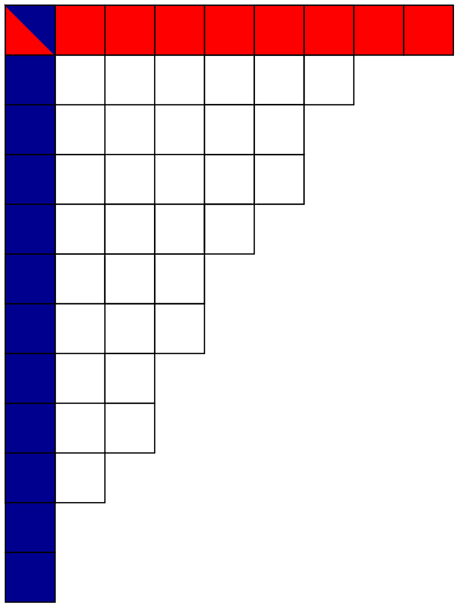
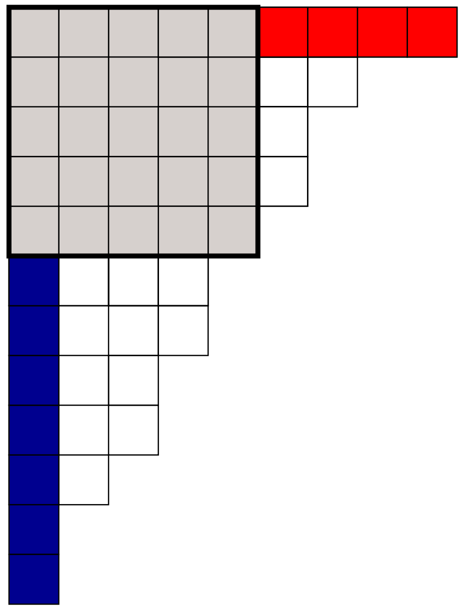
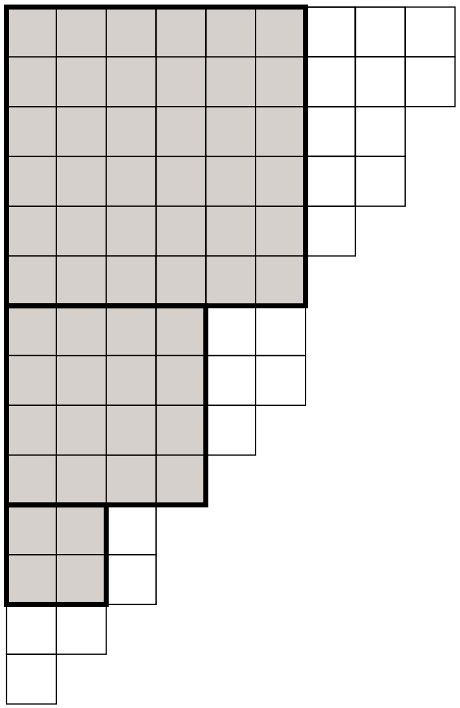




- rank$(\lambda) = 9 - 12 = -3$
- rank$(\lambda) = 4 - 7 = -3$

Definition of (k,m)-rank



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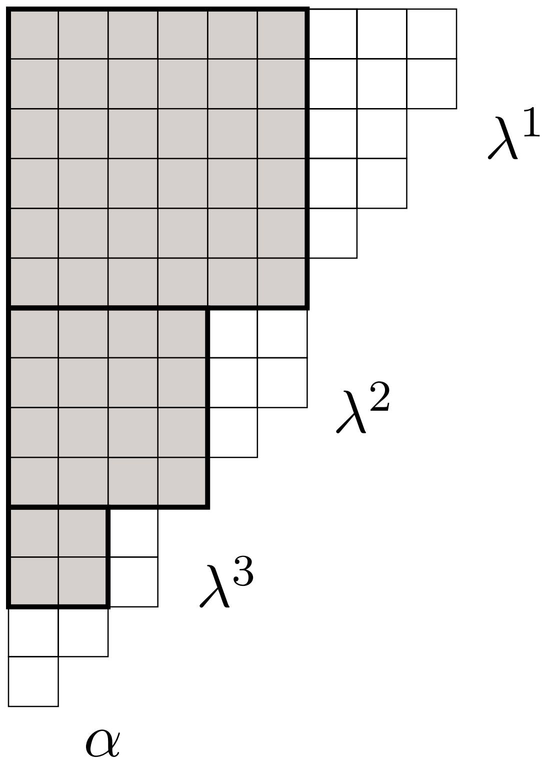

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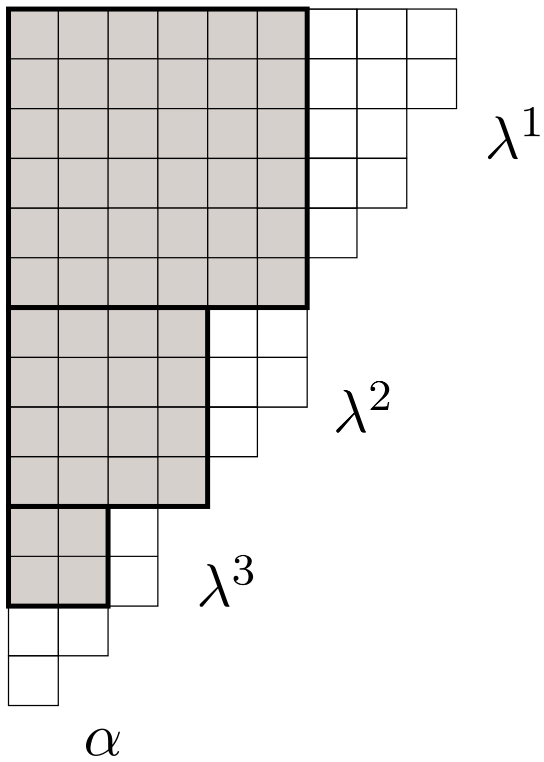
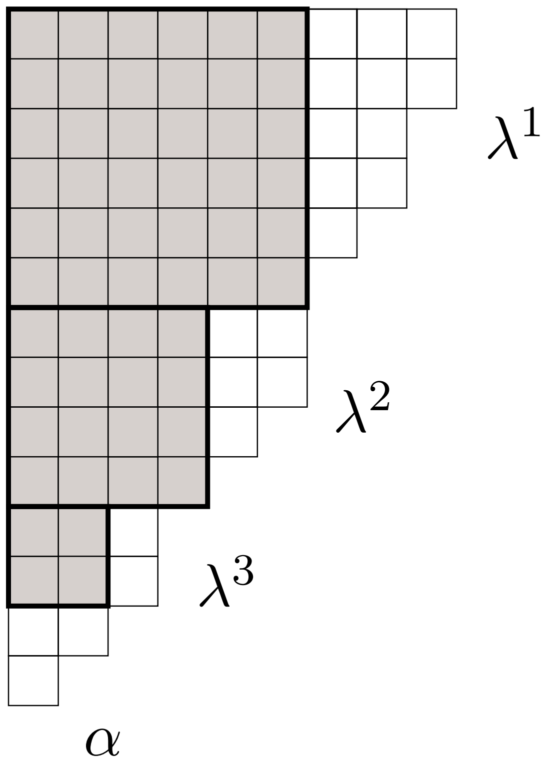


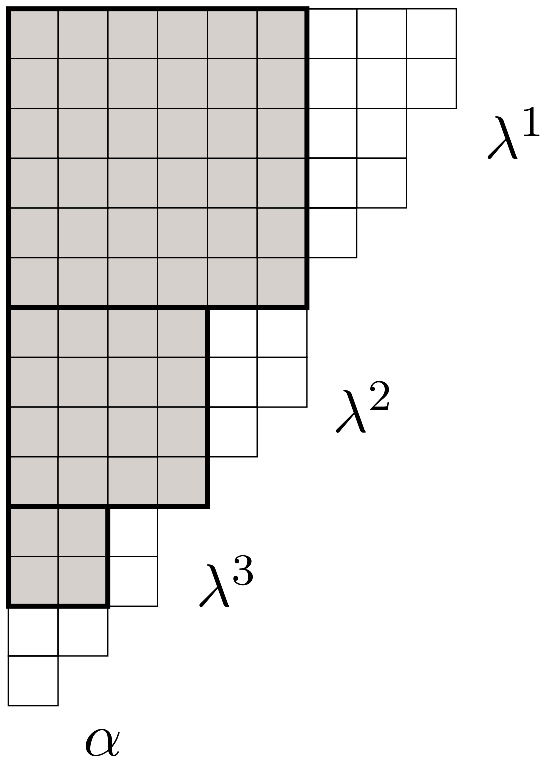
- Let N_i be the width of the ith Durfee rectangle.

Definition of (k,m)-rank



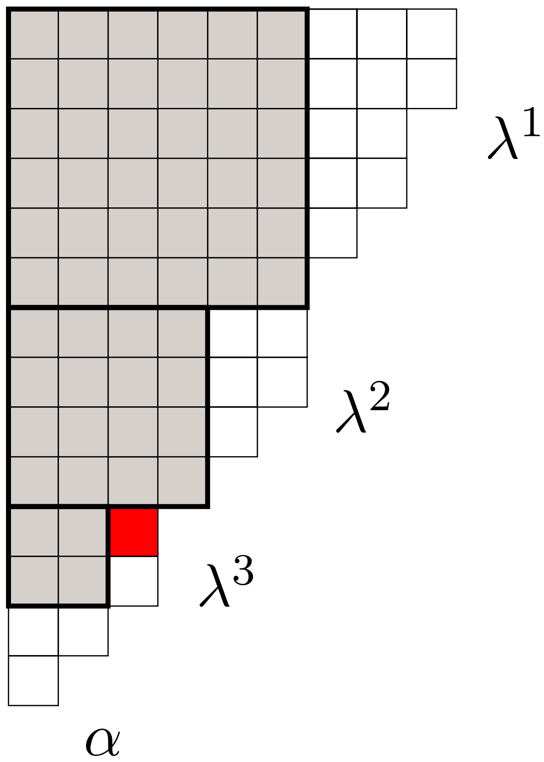
- Let N_i be the width of the ith Durfee rectangle.
- Let $p_i = N_{i-1} - N_i$.

Definition of (k,m)-rank



- Let N_i be the width of the ith Durfee rectangle.
- Let $p_i = N_{i-1} - N_i$.
- We can select rows from $\lambda^1, \lambda^2, ..., \lambda^k$.

Definition of (k,m)-rank



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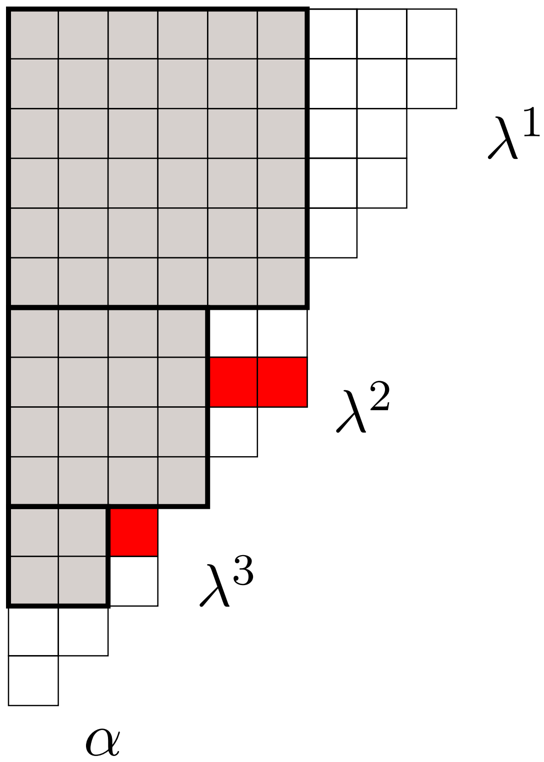

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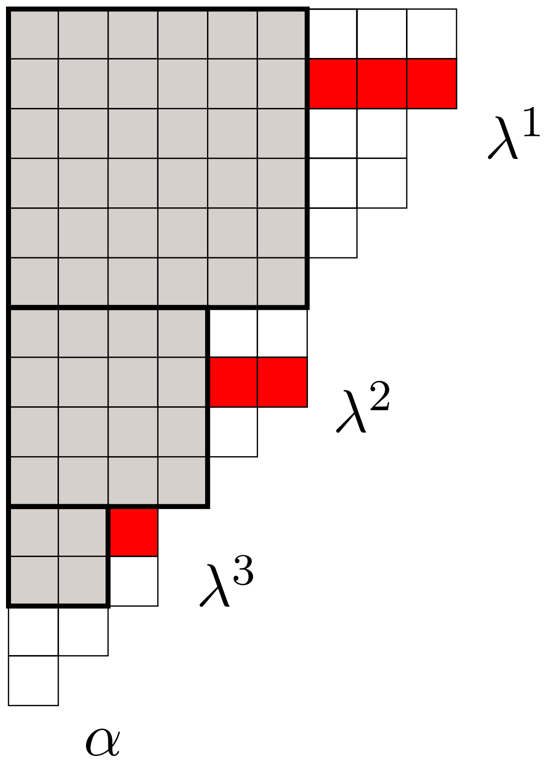

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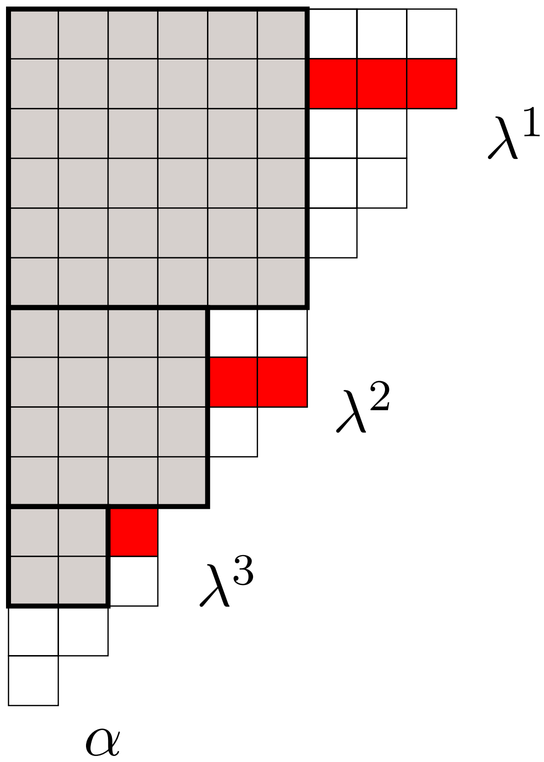

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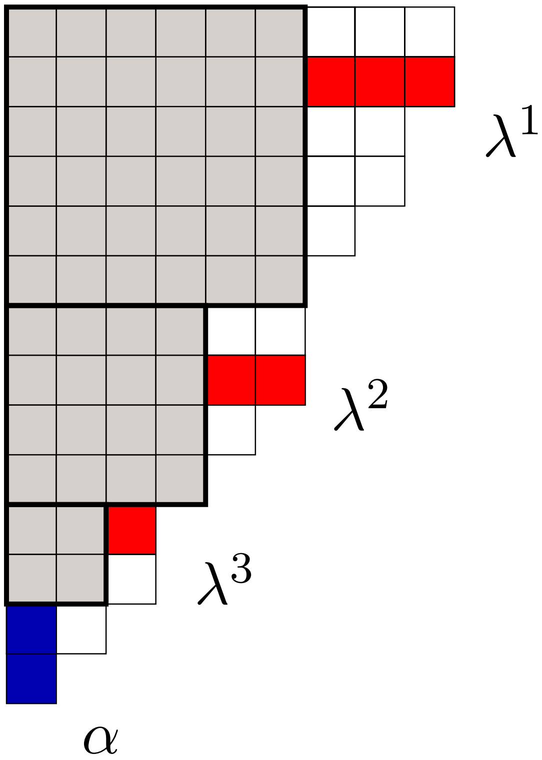
Definition of (k,m)-rank



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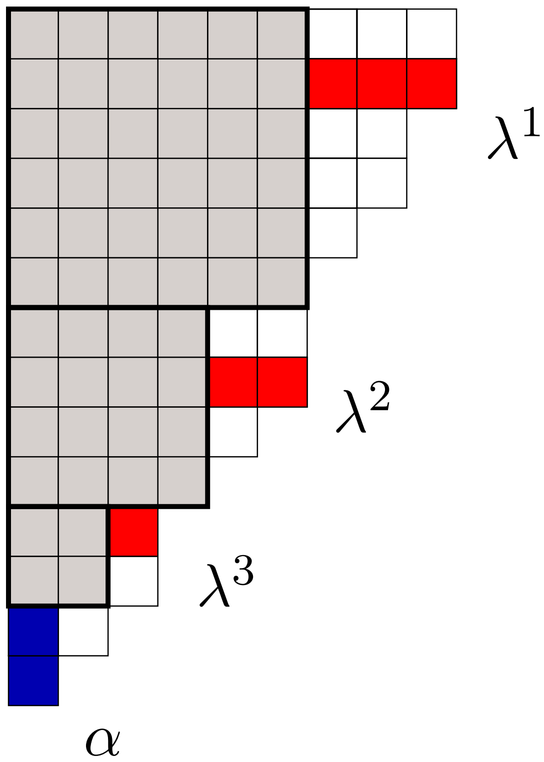

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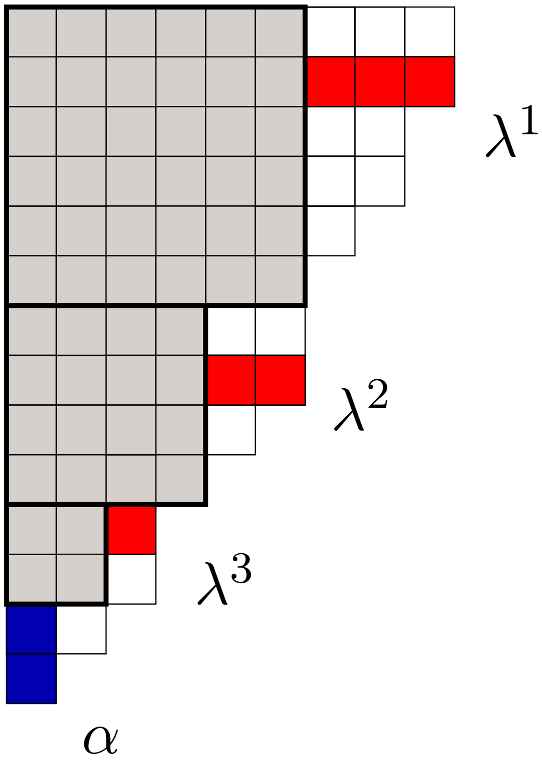
Defn. (k, m)-rank of λ: $r_{k,m}(\lambda) = A(\lambda) - \ell(\alpha)$.

Definition of (k,m)-rank



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Defn. (k, m)-rank of λ: $r_{k,m}(\lambda) = A(\lambda) - \ell(\alpha) = 4$.

Definition of (k,m)-rank

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Definition of (k,m)-rank

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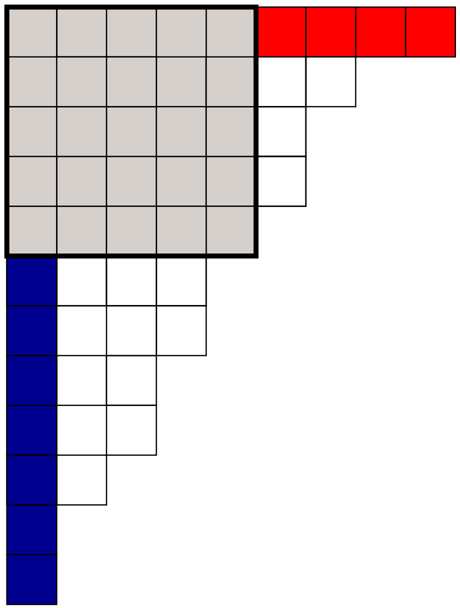
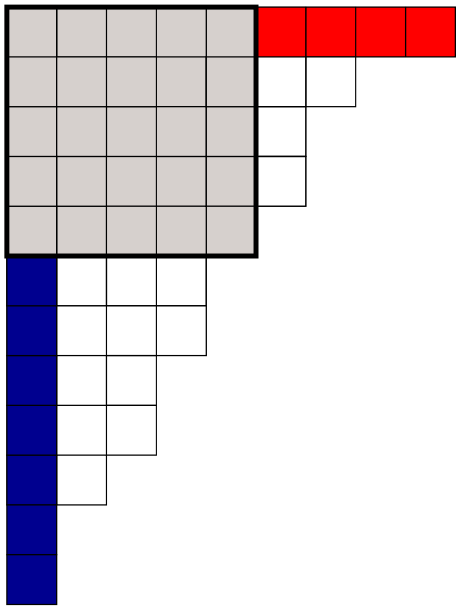


$k = 1$, $m = 0$

Definition of (k,m)-rank

Defn. (k,m)-rank of λ: $r_{k,m}(\lambda) = A(\lambda) - \ell(\alpha)$

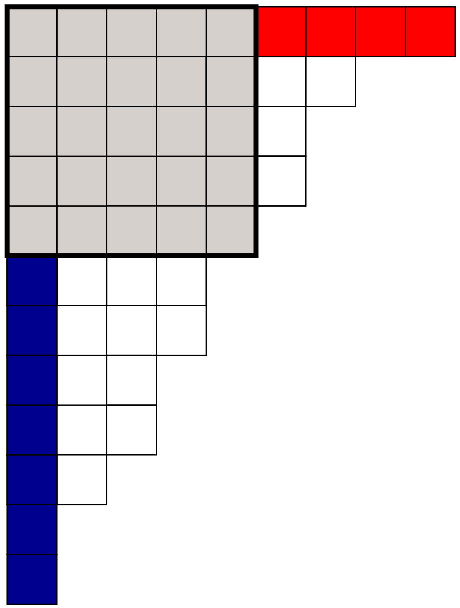


$k = 1$, $m = 0$

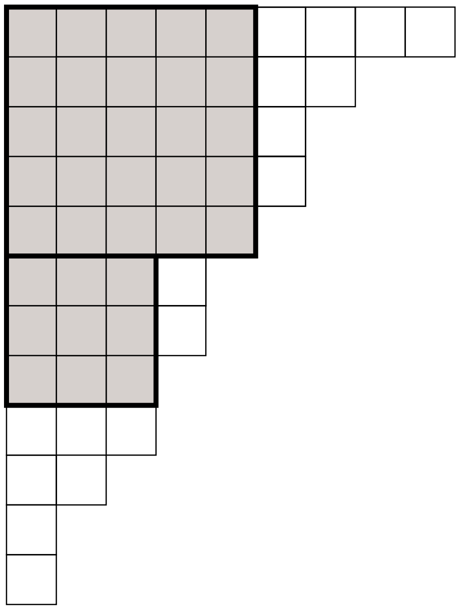
- $r_{k,m}(\lambda) = 4 - 7 = -3$

Definition of (k,m)-rank

Defn. (k, m)-rank of λ: $r_{k,m}(\lambda) = A(\lambda) - \ell(\alpha)$



$k = 1$, $m = 0$



$k = 2$, $m = 0$

- $r_{k,m}(\lambda) = 4 - 7 = -3$

Definition of (k,m)-rank

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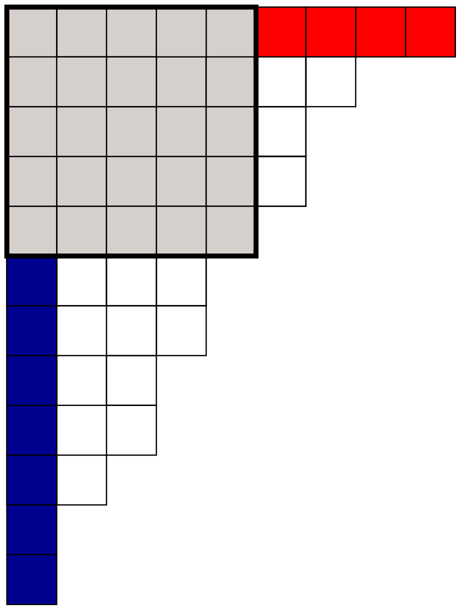

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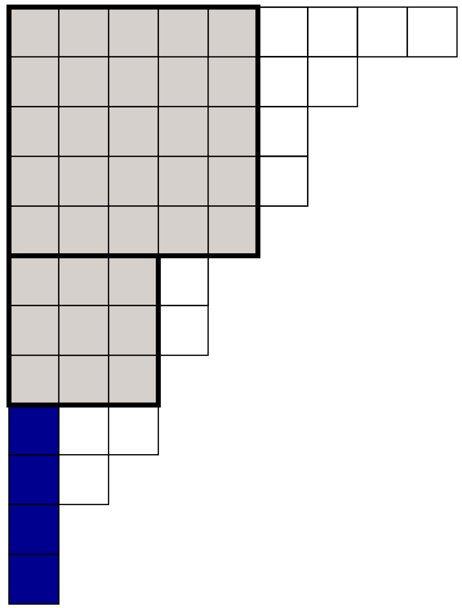

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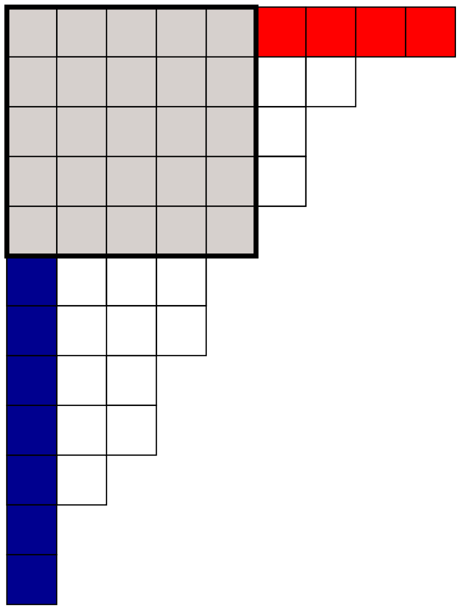

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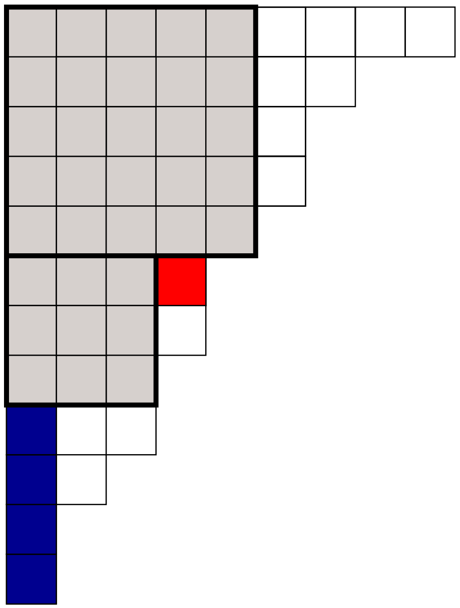

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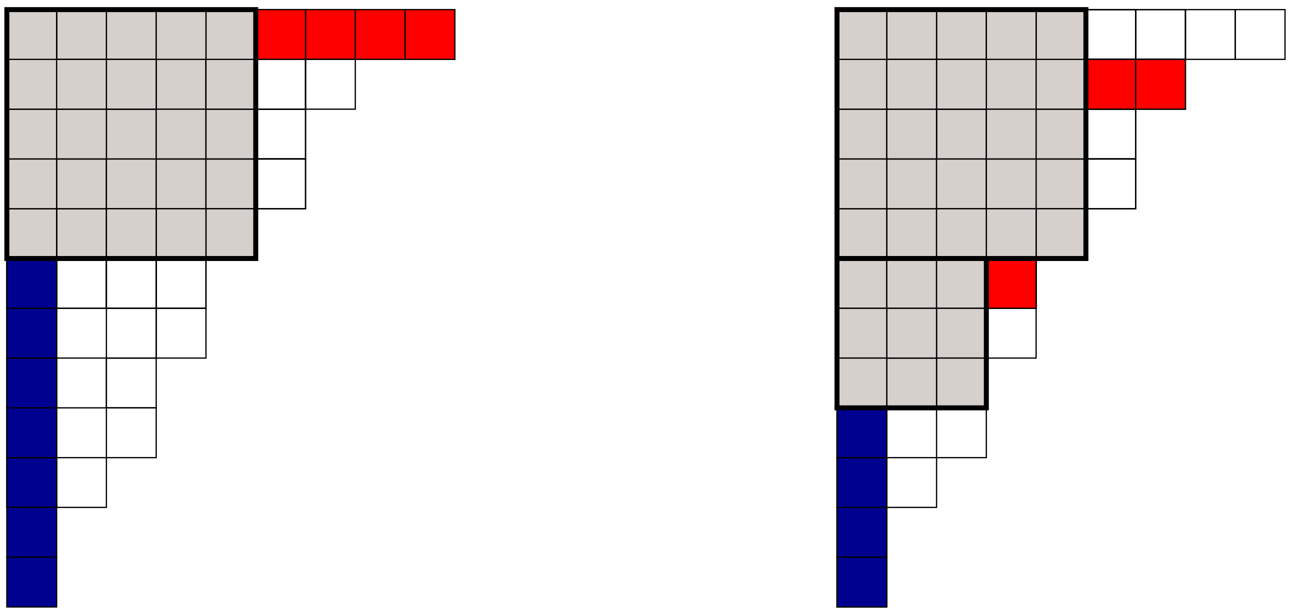

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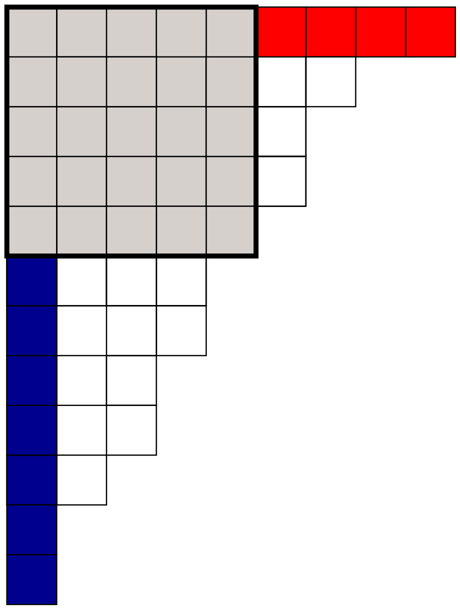

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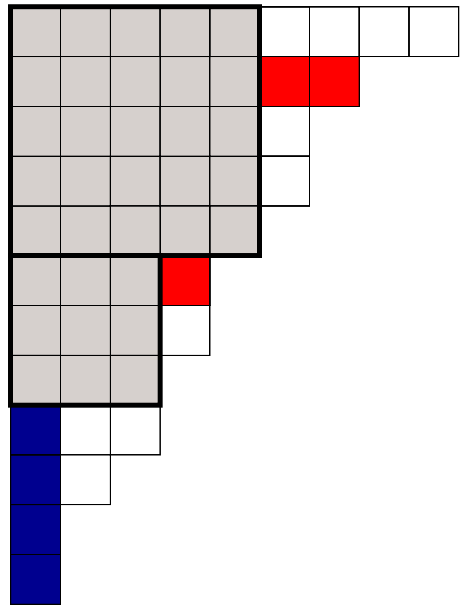

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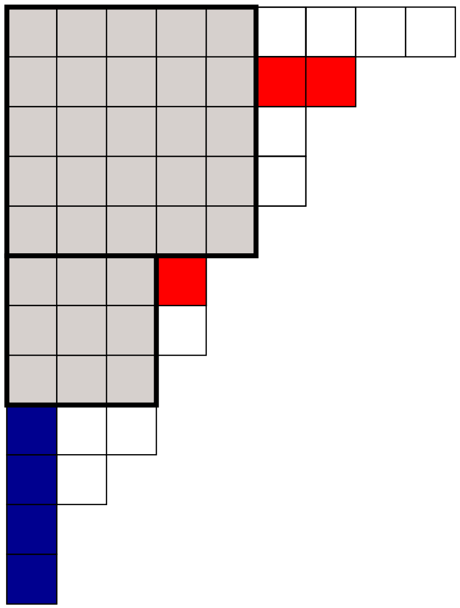

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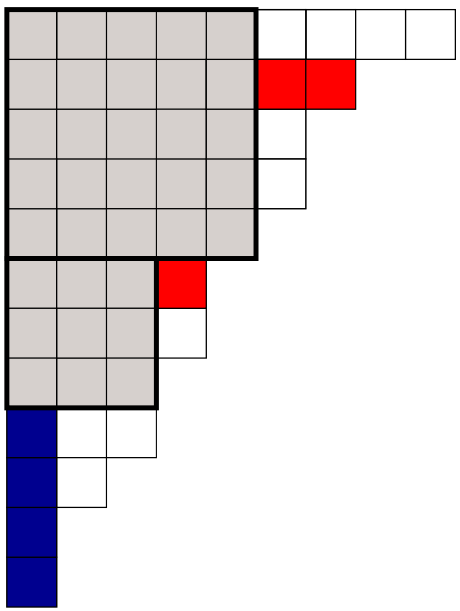

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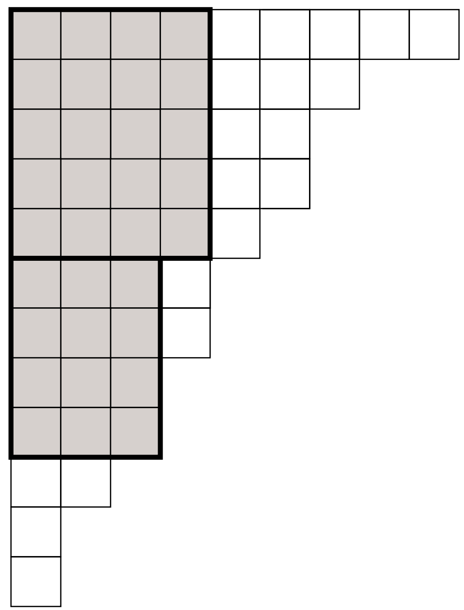

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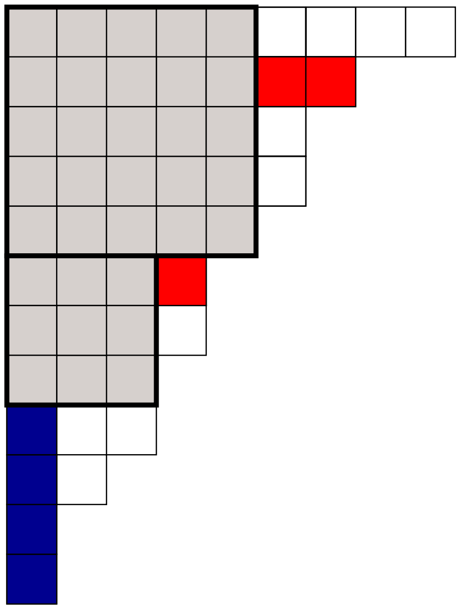

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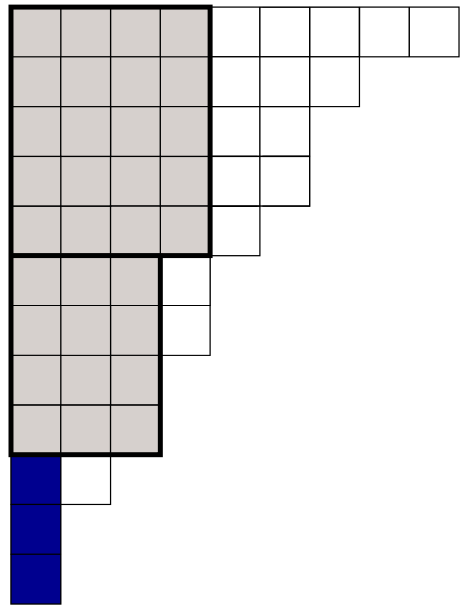

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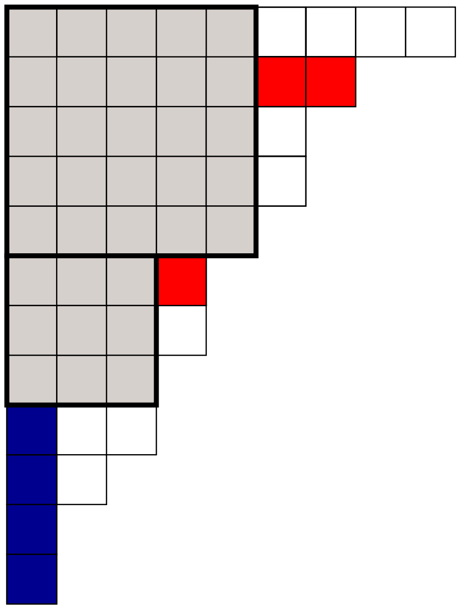

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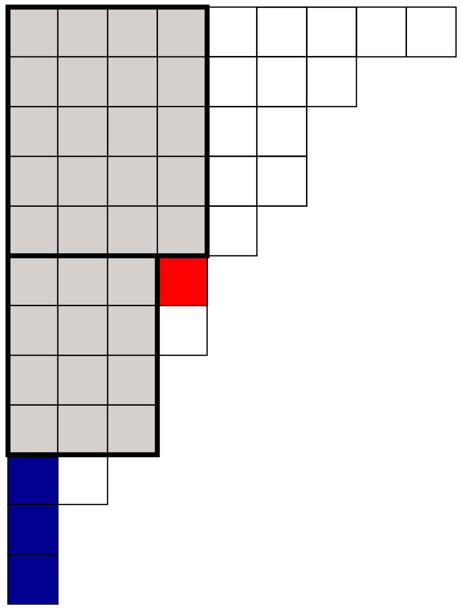

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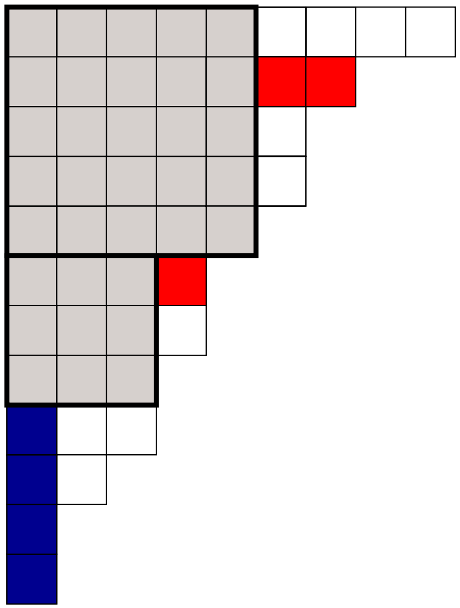

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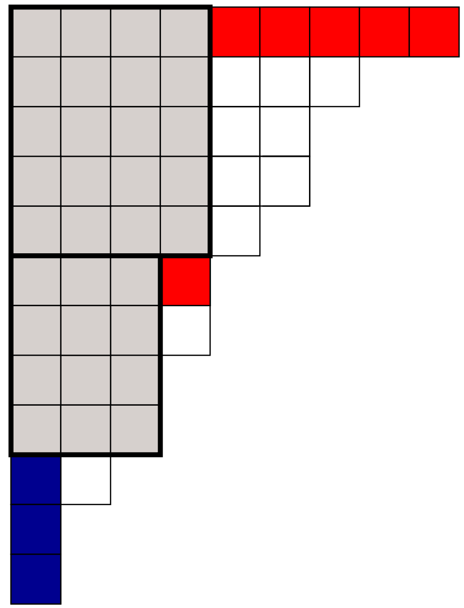

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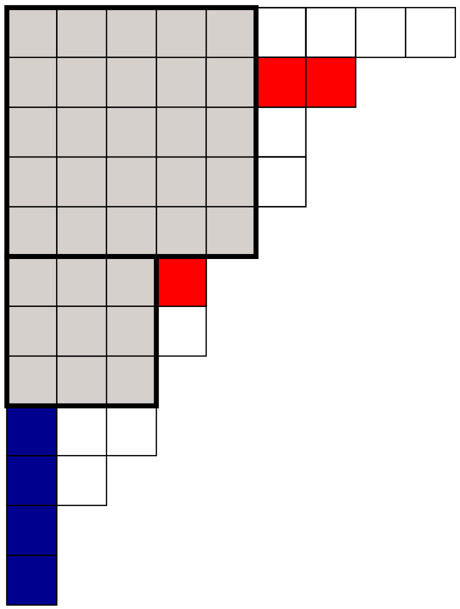

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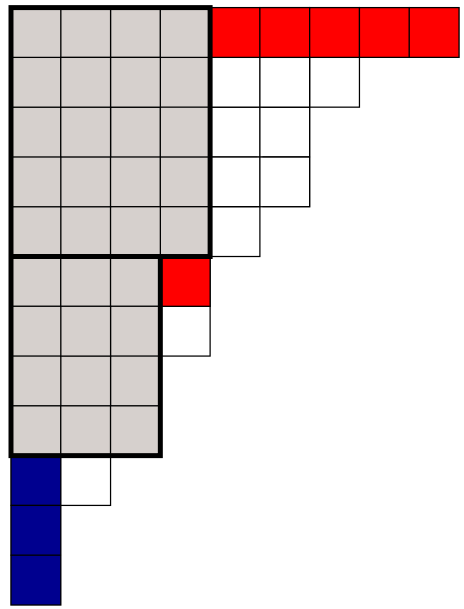

$k = 2$, $m = 1$

- $r_{k,m}(\lambda) = 1 + 5 - 3 = 3$

Observations

Let $h(n, k, m, r)$ be the number of partitions of n with (k, m)-rank equal to r.

- when $m > 0$, we always have k successive m-Durfee rectangles

$$h(n, k, m, \leq -r-1) + h(n, k, m, \geq -r) = p(n)$$

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$$h(n,k,m,\leq -r-1) + h(n,k,m,\geq -r) = p(n)$$

• when $m = 0$, we do not always have k successive m-Durfee rectangles (i.e. Durfee squares)

$$h(n,k,0,\leq -r-1) + h(n,k,0,\geq -r) = p(n) - q(n)$$

where $q(n)$ is the number of partitions of n with at most $(k-1)$ Durfee squares

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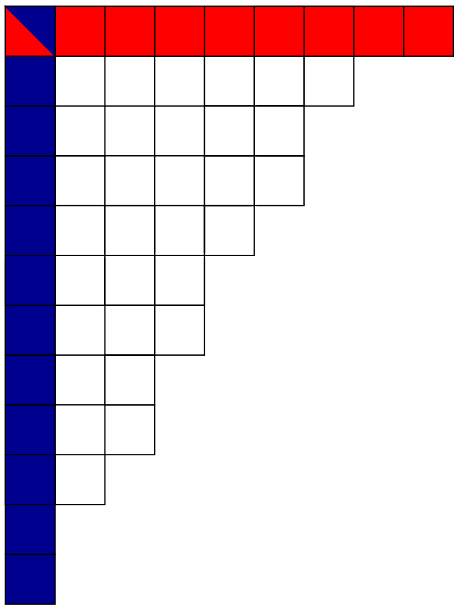

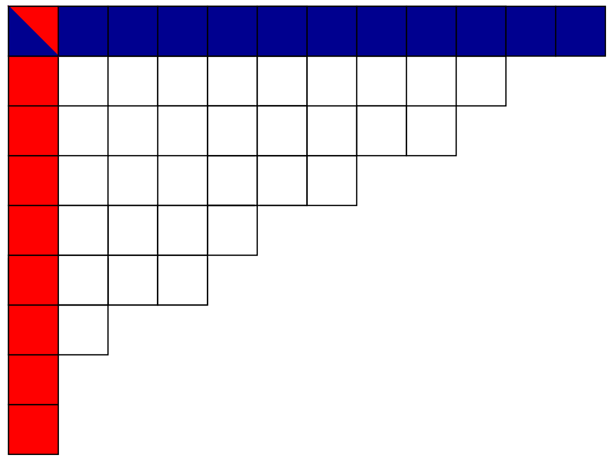

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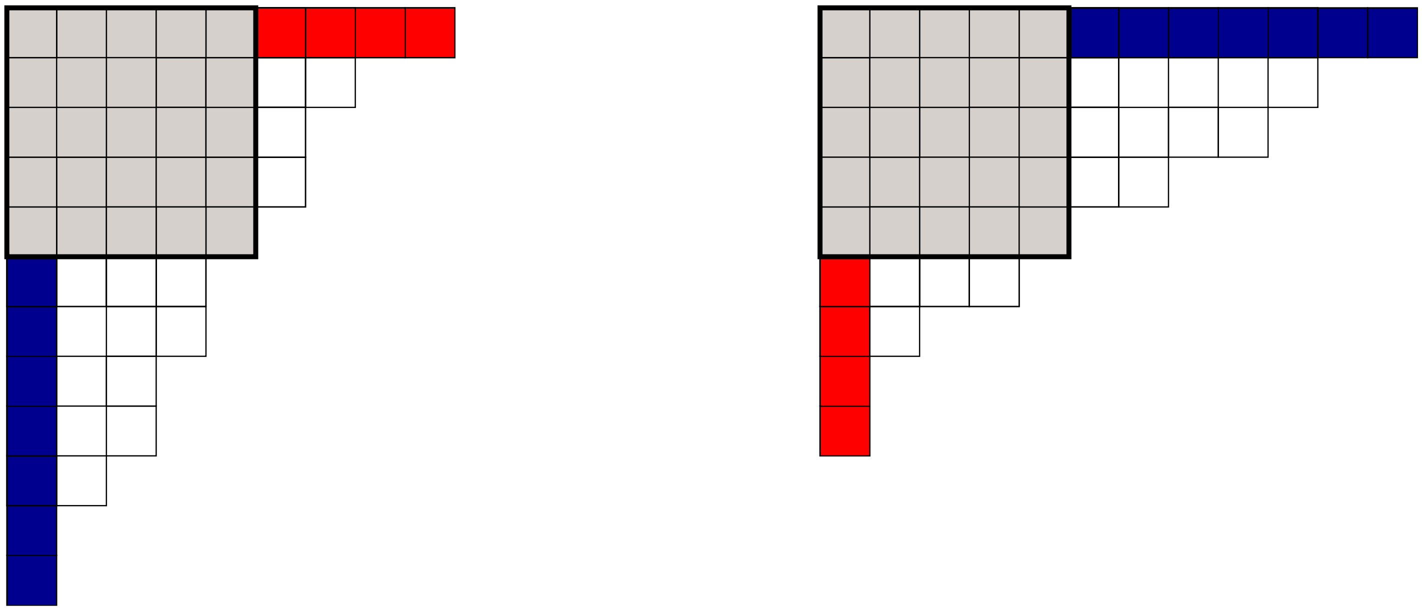

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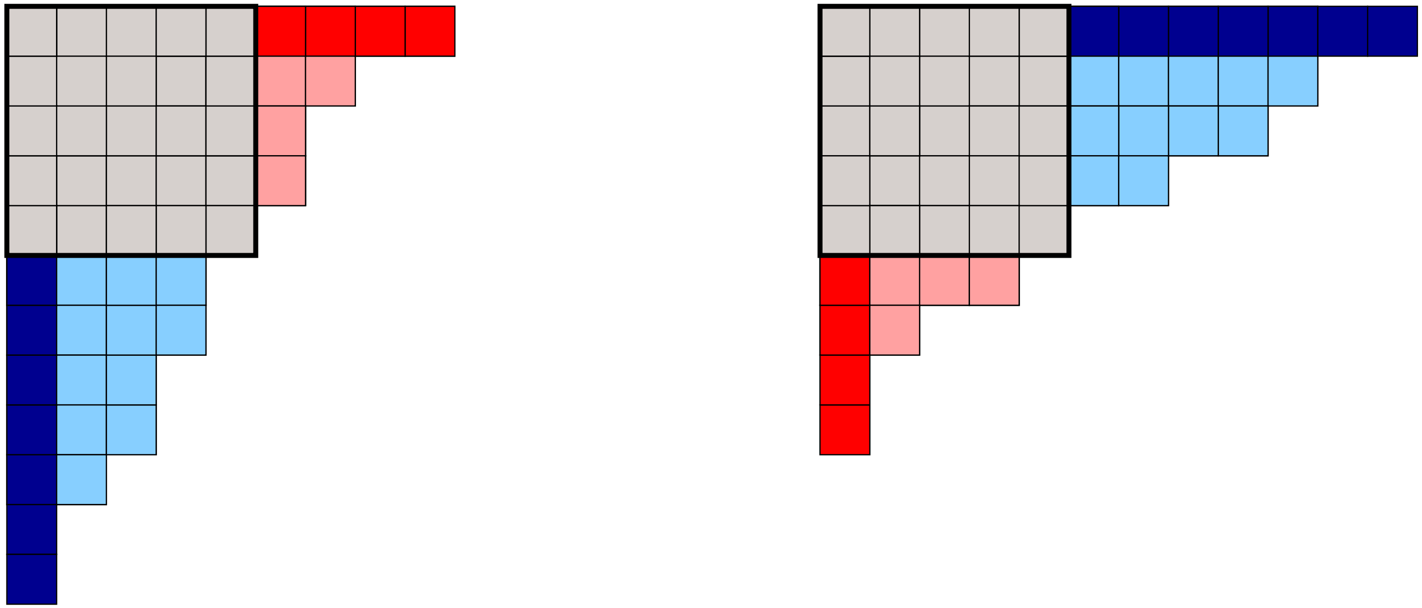

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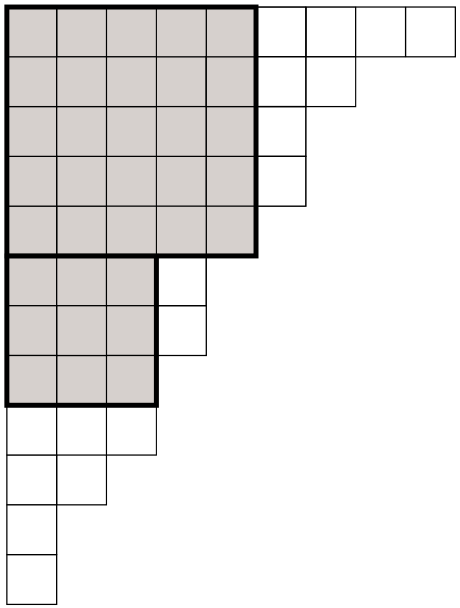

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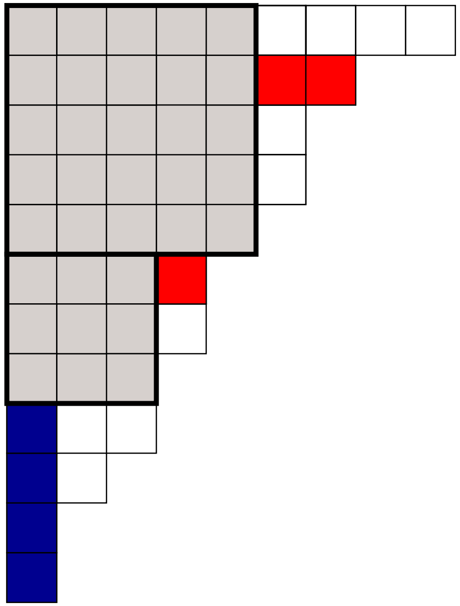

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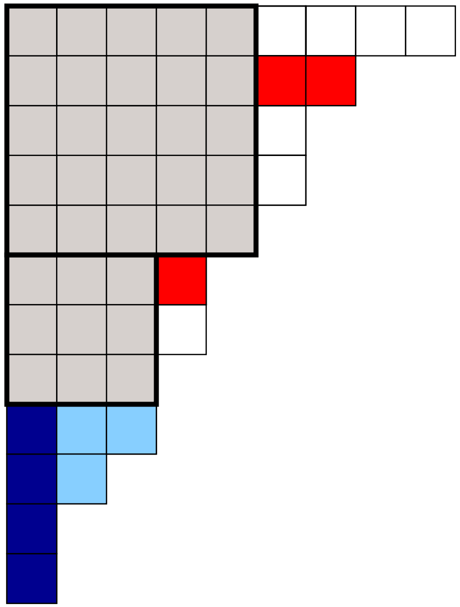

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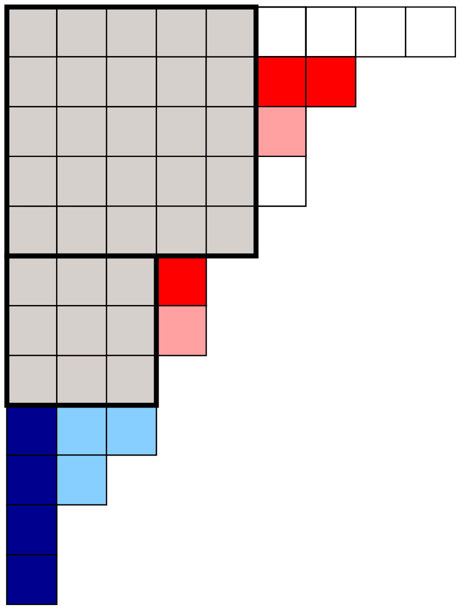

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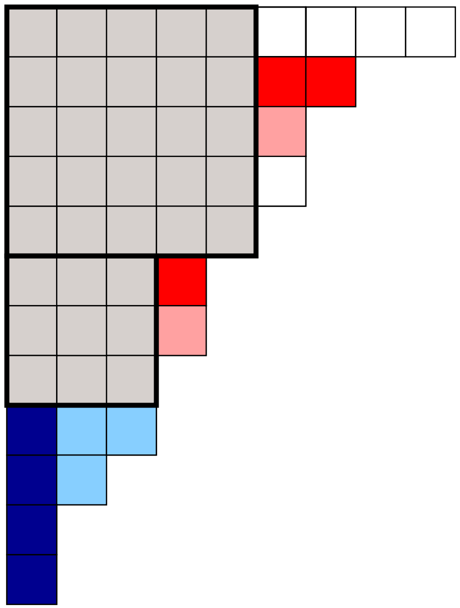

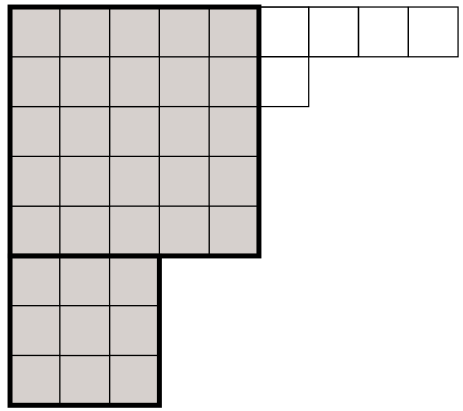

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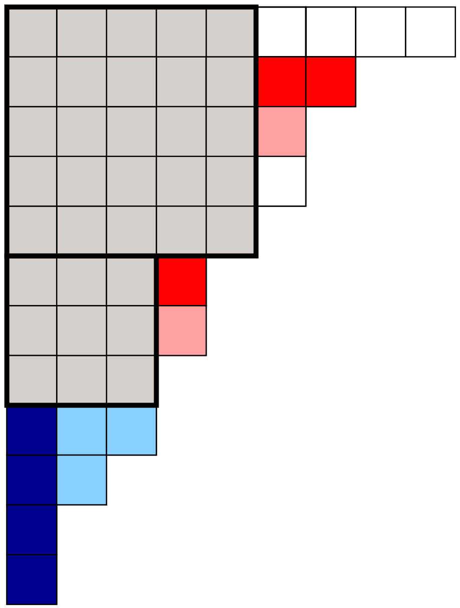

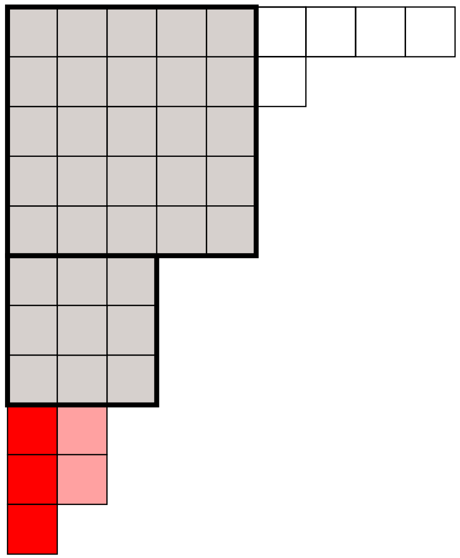

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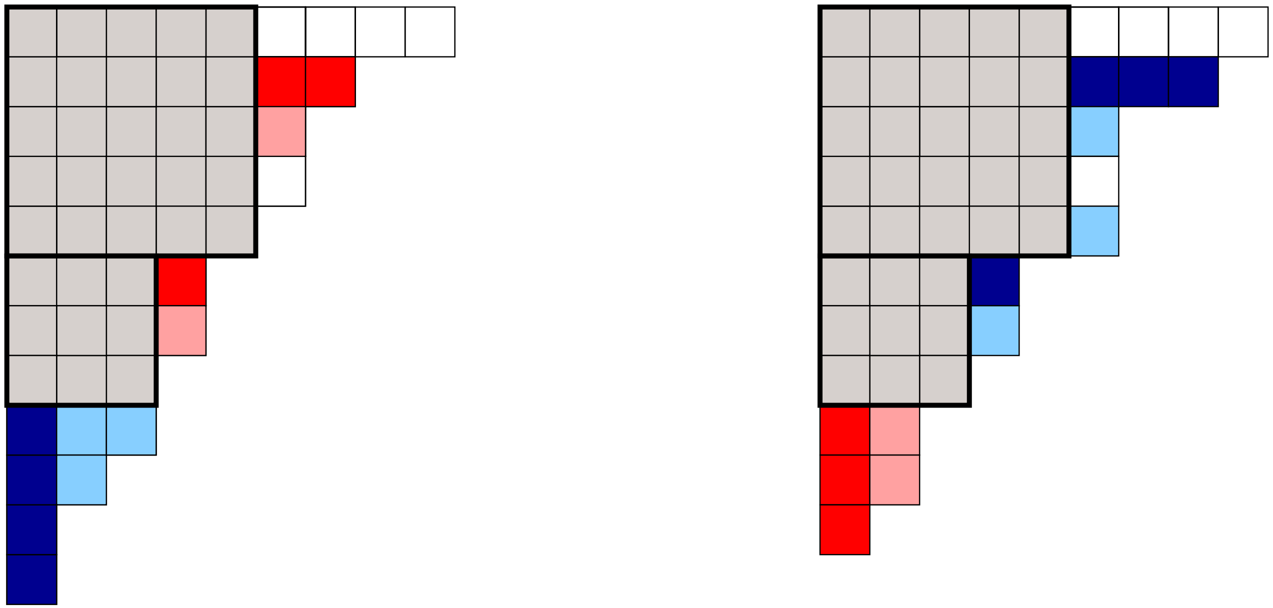

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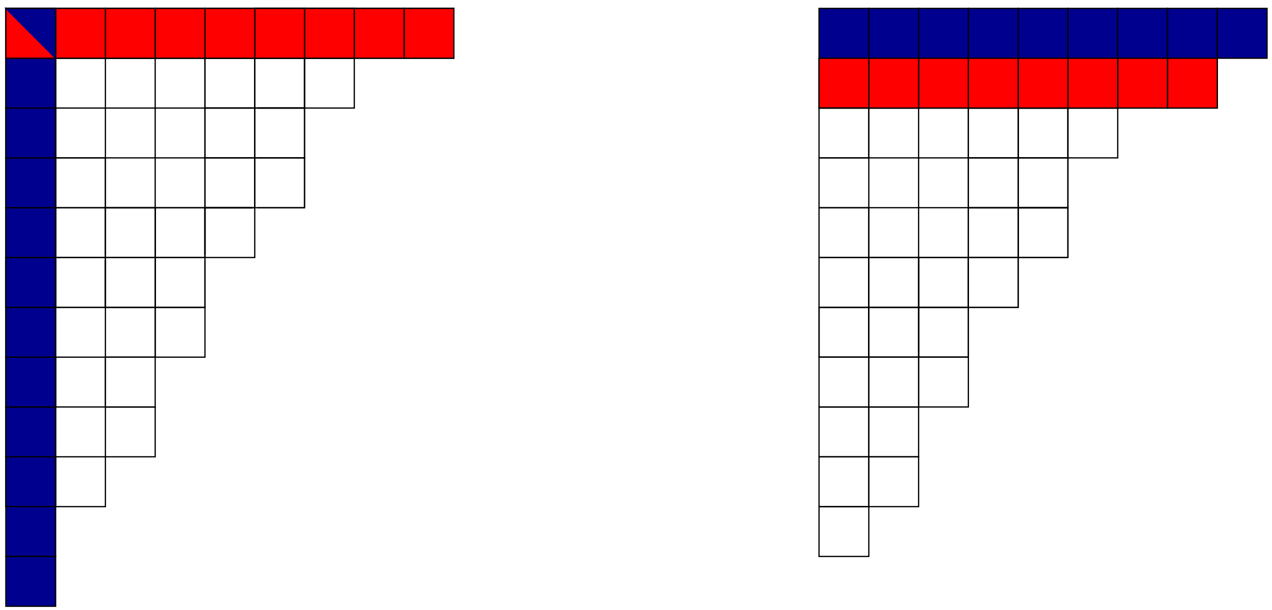

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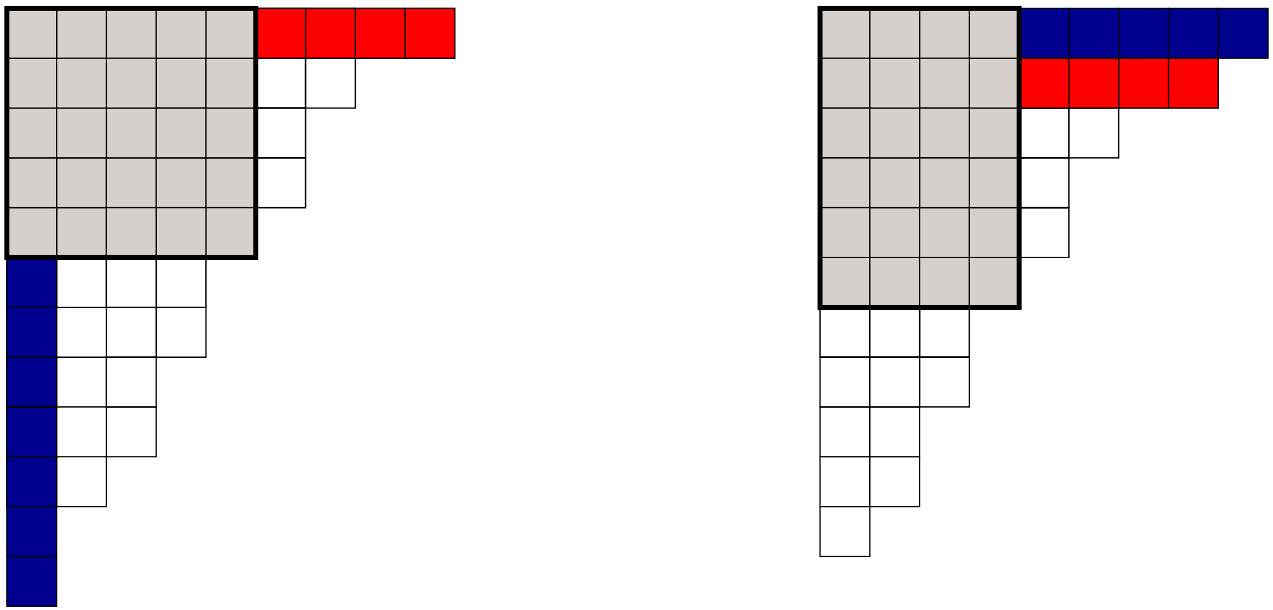

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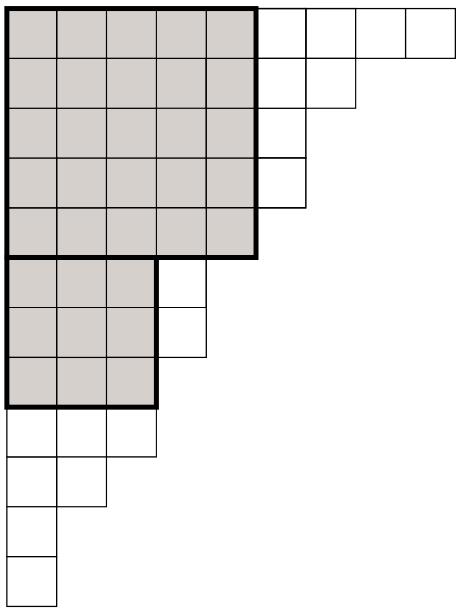

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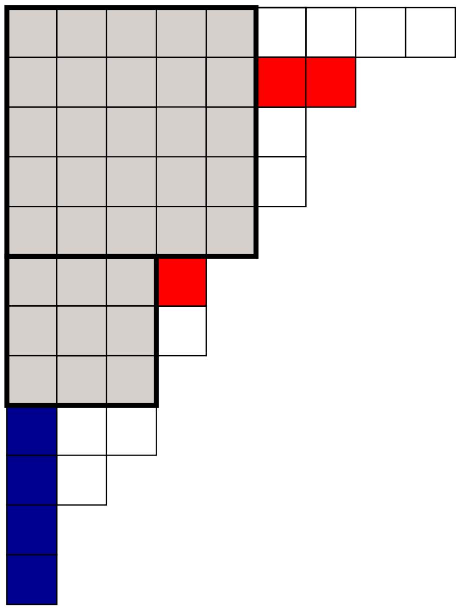

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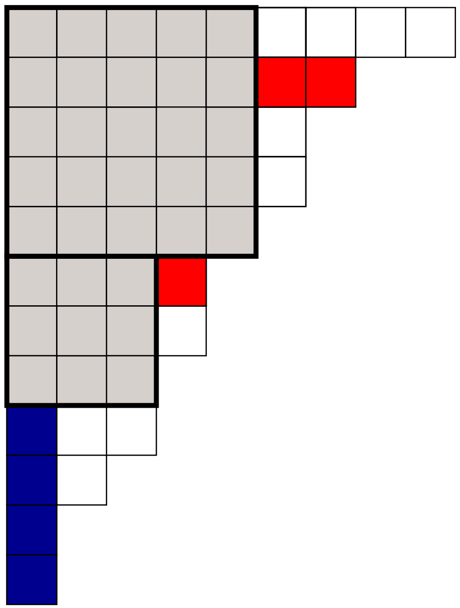

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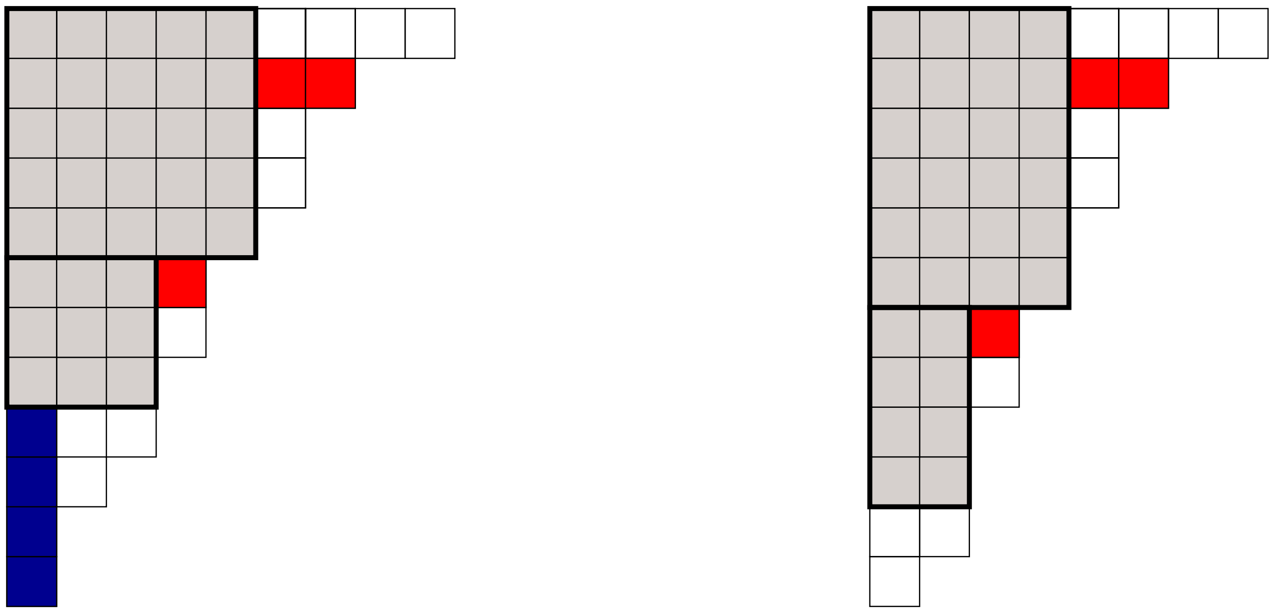

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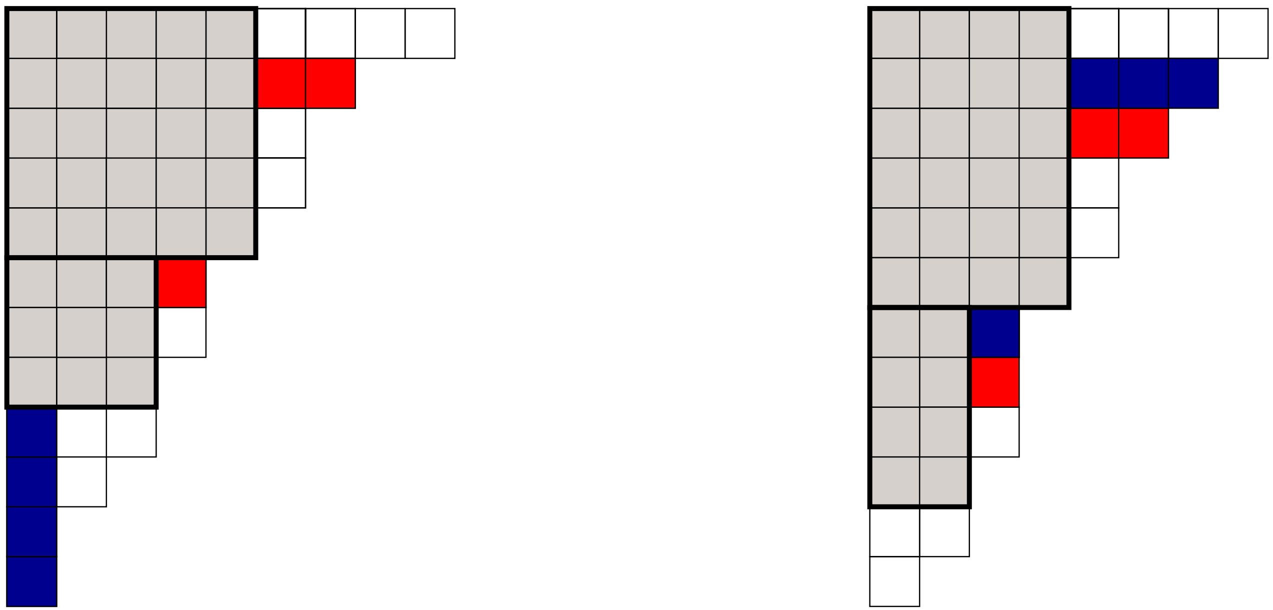

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A word about the proofs

For each bijection we have:

- for $k = 1$ and $k = 2$:

  exact formulas to describe the bijection,

- in general:

  a description in terms of insertion.

The proof of the lemma that describes insertion gives a recursive algorithm for insertion and therefore for these bijections.

Symmetries

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- and our observations ...

for $m > 0$, $h(n, k, m, \leq -r - 1) + h(n, k, m, \geq -r) = p(n)$

for $m = 0$, $h(n, k, 0, \leq -r - 1) + h(n, k, 0, \geq -r) = p(n) - q(n)$

A bit of algebra...

- Second symmetry and observation imply:

$$\sum_{(k,m)-\mathrm{rank}(\lambda)\leq -r} q^{|\lambda|} = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \sum_{j=0}^{\infty} (-1)^{j-1} q^{jr+jmk+j^2k+\frac{j(j-1)}{2}}$$

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- With first symmetry and observation, this gives:

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} =$$

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(j+1)(2k+1)}{2}-kj}$$

Q.E.D.

This is the generalized Schur's identity and therefore establishes the generalized Rogers-Ramanujan:

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} = \prod_{n \not\equiv 0, \pm k \pmod{2k+1}} \frac{1}{1-q^n}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$.

Connections

- Bressoud and Zeilberger gave a bijective proof of Rogers-Ramanujan based on the involution principle.
In some cases our second bijection, acts similarly to one of their bijections.

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In some cases our second bijection, acts similarly to one of their bijections.

- Garvan gave an alternate generalization of Dyson's rank, called k-rank.
We can prove bijectively that his $(k-1)$-rank and our $(k,0)$-rank have the same distribution (though they are not the same statistic).

Further questions

- Andrews showed that for $1 \le a \le k$:

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_a+\cdots+N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} = \prod_{n \not\equiv 0, \pm a \pmod{2k+1}} \frac{1}{1-q^n}$$

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Question.

Can our proof be extended to prove these identities?

Further questions

- Ramanujan showed:

$$p(5n+4) \equiv 0 \pmod 5$$
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- rank proves this combinatorially

(conjecture: Dyson, proof: Atkin and Swinnerton-Dyer)

Question.
Are there some n and j such that the partitions of n with at least k Durfee squares are divided into equinumerous classes (mod j) by $(k, 0)$-rank?

Further questions

- Berkovich and Garvan define a version of Dyson's map for 2-modular diagrams of partitions whose odd parts are distinct and use it to prove Gauss' identity.

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• Berkovich and Garvan define a version of Dyson's map for 2-modular diagrams of partitions whose odd parts are distinct and use it to prove Gauss' identity.

Question.
Can we find versions of our bijection for some family of weighted Young diagrams, and what identities could we prove with these maps?

Further questions

- iterating Dyson's map gives a bijection between partitions into odd parts and partitions into distinct parts (Pak).

Further questions

- iterating Dyson's map gives a bijection between partitions into odd parts and partitions into distinct parts (Pak).

Question.
Can we find an analogue of this result for partitions with at least k Durfee squares?