# A four-parameter partition identity

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George Andrews proved the following identity:

$$\sum_{\lambda \in \mathrm{Par}} r^{\theta(\lambda)} s^{\theta(\lambda')} q^{|\lambda|} \quad = \quad \prod_{j=1}^{\infty} \frac{(1 + rsq^{2j-1})}{(1-q^{4j})(1-r^2q^{4j-2})(1-s^2q^{4j-2})}$$

where $\mathrm{Par}$ denotes the set of all partitions,

$|\lambda|$ denotes the size (sum of the parts) of $\lambda$,

$\theta(\lambda)$ denotes the number of odd parts in the partition $\lambda$, and

$\theta(\lambda')$ denotes the number of odd parts in the conjugate of $\lambda$.

We will use the following weight functions on the set of all partitions:

$$\begin{aligned}
\alpha(\lambda) &= \sum \lceil \lambda_{2i-1}/2 \rceil \\
\beta(\lambda) &= \sum \lfloor \lambda_{2i-1}/2 \rfloor \\
\gamma(\lambda) &= \sum \lceil \lambda_{2i}/2 \rceil \\
\delta(\lambda) &= \sum \lfloor \lambda_{2i}/2 \rfloor .
\end{aligned}$$

Also, let $a$, $b$, $c$, $d$ be (commuting) indeterminates, and define

$$w(\lambda) = a^{\alpha(\lambda)} b^{\beta(\lambda)} c^{\gamma(\lambda)} d^{\delta(\lambda)}.$$

These were first introduced by Stanley.

Consider the following diagram for $\lambda$:

| | | | | |
|---|---|---|---|---|
| $a$ | $b$ | $a$ | $b$ | $a$ |
| $c$ | $d$ | $c$ | $d$ | |
| $a$ | $b$ | $a$ | $b$ | |
| $c$ | $d$ | $c$ | | |
| $a$ | $b$ | | | |

Then,

$\alpha(\lambda)$ is the number of $a$'s in the diagram,

$\beta(\lambda)$ is the number of $b$'s in the diagram,

$\gamma(\lambda)$ is the number of $c$'s in the diagram, and

$\delta(\lambda)$ is the number of $d$'s in the diagram.

Moreover, $w(\lambda)$ is the product of the entries of the diagram.

**Theorem 1.**

$$\sum_{\lambda \in \mathrm{Par}} w(\lambda) = \prod_{j=1}^{\infty} \frac{(1 + a^j b^j c^j d^{j-1})(1 + a^j b^{j-1} c^{j-1} d^{j-1})}{(1 - a^j b^j c^j d^j)(1 - a^j b^j c^{j-1} d^{j-1})(1 - a^j b^{j-1} c^j d^{j-1})}$$

## Theorem 1.

$$\sum_{\lambda \in \mathrm{Par}} w(\lambda) = \prod_{j=1}^{\infty} \frac{(1+a^j b^j c^j d^{j-1})(1+a^j b^{j-1} c^{j-1} d^{j-1})}{(1-a^j b^j c^j d^j)(1-a^j b^j c^{j-1} d^{j-1})(1-a^j b^{j-1} c^j d^{j-1})}$$

From this we obtain Andrews' result by noting that

- $|\lambda| = \alpha(\lambda) + \beta(\lambda) + \gamma(\lambda) + \delta(\lambda)$,
- $\theta(\lambda) = \alpha(\lambda) - \beta(\lambda) + \gamma(\lambda) - \delta(\lambda)$,
- $\theta(\lambda') = \alpha(\lambda) + \beta(\lambda) - \gamma(\lambda) - \delta(\lambda)$.

# Proof: Plan

$$\sum_{\lambda \in \mathrm{Par}} w(\lambda) = \prod_{j=1}^{\infty} \frac{(1 + a^j b^j c^j d^{j-1})(1 + a^j b^{j-1} c^{j-1} d^{j-1})}{(1 - a^j b^j c^j d^j)(1 - a^j b^{j-1} c^j d^{j-1})(1 - a^j b^j c^{j-1} d^{j-1})}$$

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$$RB = \{\lambda \in \mathrm{Par} : \lambda_{2i-1} - \lambda_{2i} \leq 1\}$$

$G = \{\lambda \in \mathrm{Par} : \lambda'$ has only odd parts, each of which is repeated an even number of times$\}$

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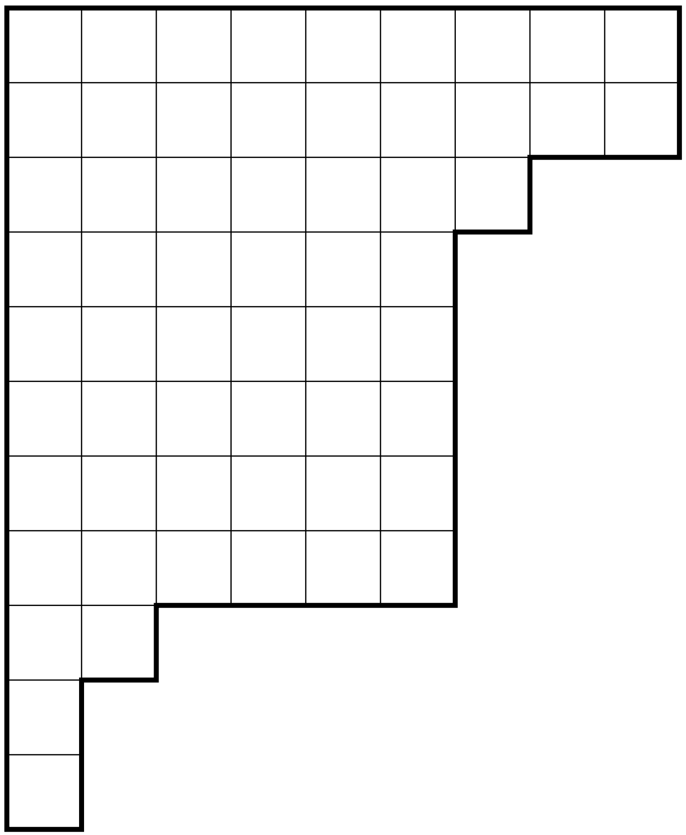
- bijection: $RB \times G \longleftrightarrow \mathrm{Par}$

# Proof: Generating Function for $RB$

$$RB = \{\lambda \in \mathrm{Par} : \lambda_{2i-1} - \lambda_{2i} \leq 1\}$$



- $\lambda \in RB$

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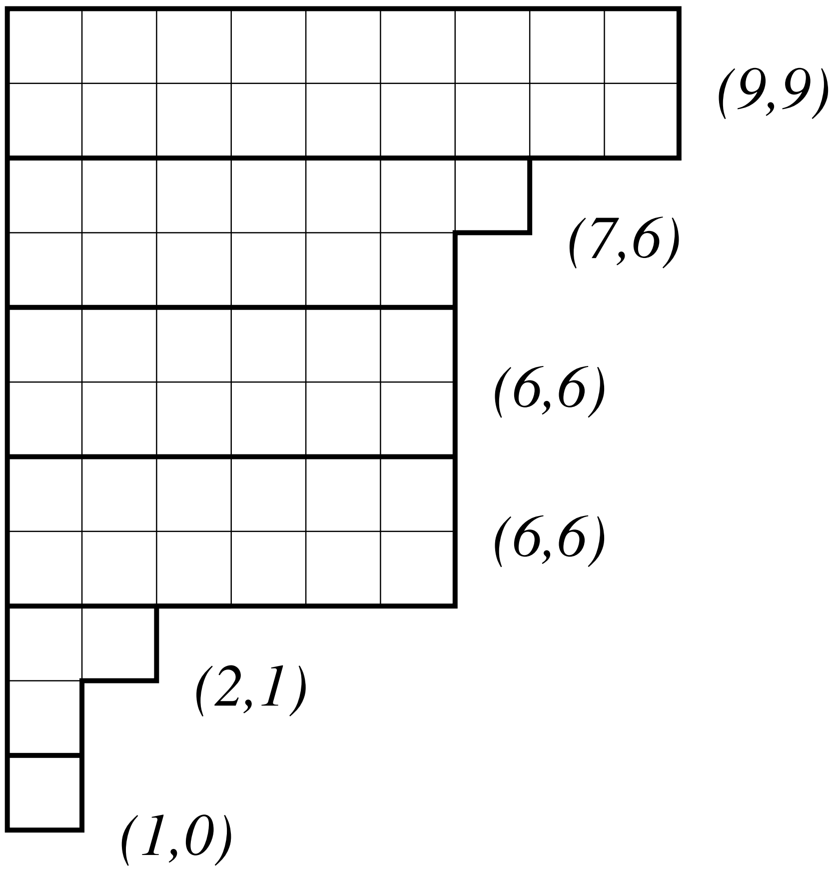


- $\lambda \in RB$
- blocks of height two have parts equal or differing by at most 1

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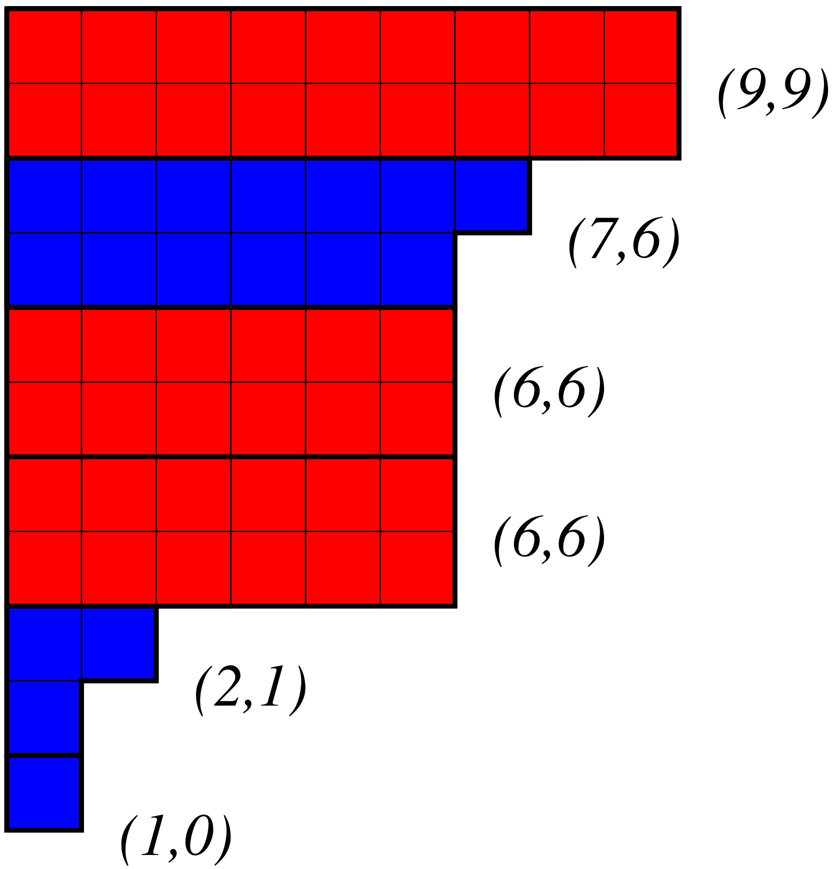


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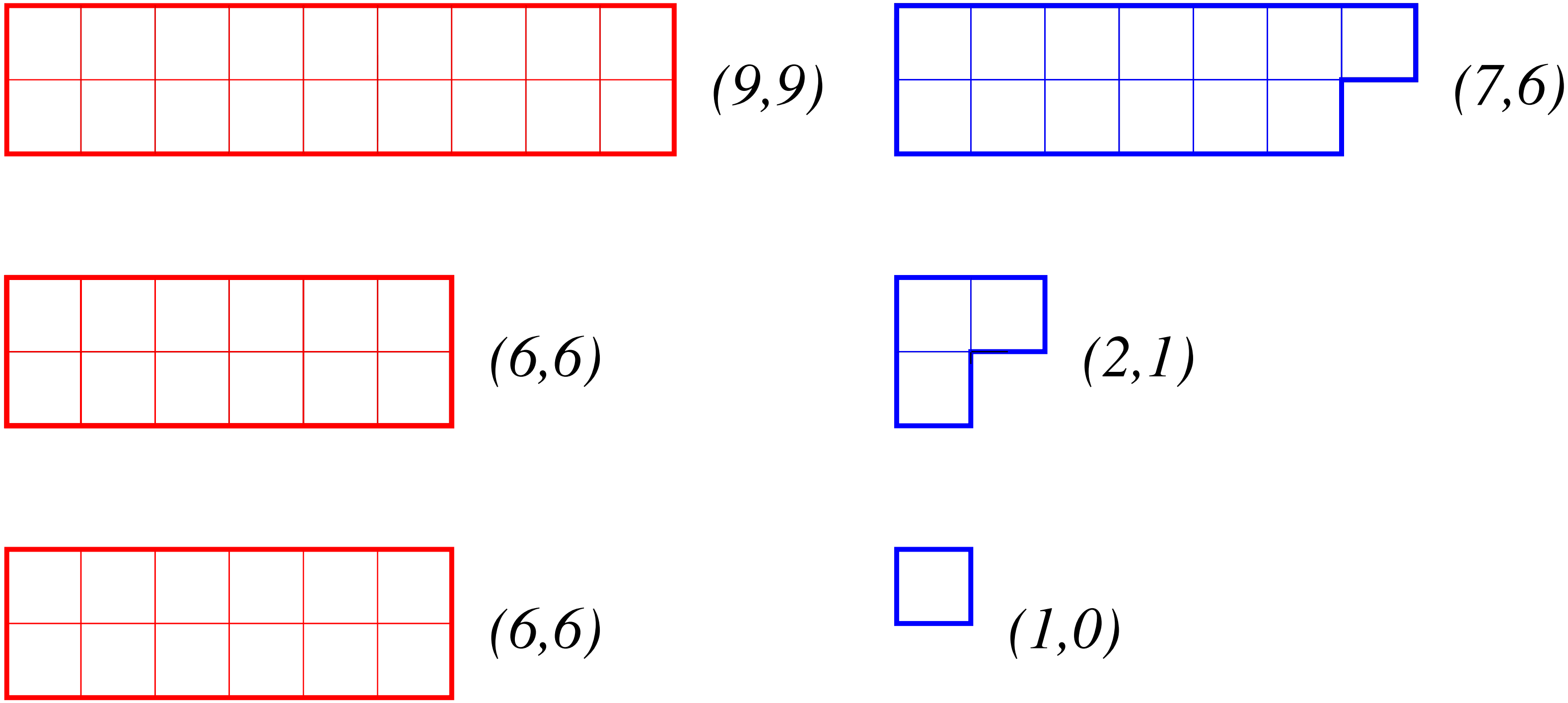
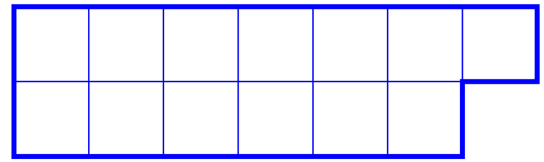


# Proof: Generating Function for $RB$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $a$ | $b$ | $a$ | $b$ | $a$ | $b$ | $a$ | $b$ | $a$ |
| $c$ | $d$ | $c$ | $d$ | $c$ | $d$ | $c$ | $d$ | $c$ |

(9,9)

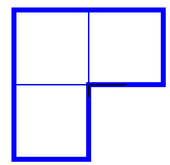


(7,6)

$a^5 b^4 c^5 d^4$

| | | | | | |
|---|---|---|---|---|---|
| $a$ | $b$ | $a$ | $b$ | $a$ | $b$ |
| $c$ | $d$ | $c$ | $d$ | $c$ | $d$ |

(6,6)



(2,1)

$a^3 b^3 c^3 d^3$

| | | | | | |
|---|---|---|---|---|---|
| $a$ | $b$ | $a$ | $b$ | $a$ | $b$ |
| $c$ | $d$ | $c$ | $d$ | $c$ | $d$ |

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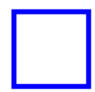

(1,0)

$a^3 b^3 c^3 d^3$

- weights: $a^j b^j c^j d^j$

  $a^j b^{j-1} c^j d^{j-1}$

# Proof: Generating Function for $\color{red}{R}\color{blue}{B}$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $a$ | $b$ | $a$ | $b$ | $a$ | $b$ | $a$ | $b$ | $a$ |
| $c$ | $d$ | $c$ | $d$ | $c$ | $d$ | $c$ | $d$ | $c$ |

(9,9)

$a^5 b^4 c^5 d^4$

| | | | | | |
|---|---|---|---|---|---|
| $a$ | $b$ | $a$ | $b$ | $a$ | $b$ |
| $c$ | $d$ | $c$ | $d$ | $c$ | $d$ |

(6,6)

$a^3 b^3 c^3 d^3$

| | | | | | |
|---|---|---|---|---|---|
| $a$ | $b$ | $a$ | $b$ | $a$ | $b$ |
| $c$ | $d$ | $c$ | $d$ | $c$ | $d$ |

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  $a^j b^{j-1} c^j d^{j-1}$

| | | | | | | |
|---|---|---|---|---|---|---|
| $a$ | $b$ | $a$ | $b$ | $a$ | $b$ | $a$ |
| $c$ | $d$ | $c$ | $d$ | $c$ | $d$ | |

(7,6)

$a^2 b^3 c^3 d^3$

| | |
|---|---|
| $a$ | $b$ |
| $c$ | |

(2,1)

$a \ b \ c$

| |
|---|
| $a$ |

(1,0)

$a$

- weights: $a^j b^j c^j d^{j-1}$

  $a^j b^{j-1} c^{j-1} d^{j-1}$

# Proof: Generating Function for $RB$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $a$ | $b$ | $a$ | $b$ | $a$ | $b$ | $a$ | $b$ | $a$ |
| $c$ | $d$ | $c$ | $d$ | $c$ | $d$ | $c$ | $d$ | $c$ |

$(9,9)$

$a^5 b^4 c^5 d^4$

| | | | | | | |
|---|---|---|---|---|---|---|
| $a$ | $b$ | $a$ | $b$ | $a$ | $b$ | $a$ |
| $c$ | $d$ | $c$ | $d$ | $c$ | $d$ | |

$(7,6)$

$a^2 b^3 c^3 d^3$

| | | | | | |
|---|---|---|---|---|---|
| $a$ | $b$ | $a$ | $b$ | $a$ | $b$ |
| $c$ | $d$ | $c$ | $d$ | $c$ | $d$ |

$(6,6)$

$a^3 b^3 c^3 d^3$

| | |
|---|---|
| $a$ | $b$ |
| $c$ | |

$(2,1)$

$a\ b\ c$

| | | | | | |
|---|---|---|---|---|---|
| $a$ | $b$ | $a$ | $b$ | $a$ | $b$ |
| $c$ | $d$ | $c$ | $d$ | $c$ | $d$ |

$(6,6)$

$a^3 b^3 c^3 d^3$

| |
|---|
| $a$ |

$(1,0)$

$a$

- weights: $a^j b^j c^j d^j$

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- weights: $a^j b^j c^j d^{j-1}$

  $a^j b^{j-1} c^{j-1} d^{j-1}$

- may occur multiple times

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| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $a$ | $b$ | $a$ | $b$ | $a$ | $b$ | $a$ | $b$ | $a$ |
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| $a$ | $b$ | $a$ | $b$ | $a$ | $b$ |
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|---|---|
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(2,1)

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| |
|---|
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(1,0)

$a$

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- occurs at most once

# Proof: Generating Function for $RB$

- weights: $a^j b^j c^j d^j$

  $a^j b^{j-1} c^j d^{j-1}$
- may occur multiple times

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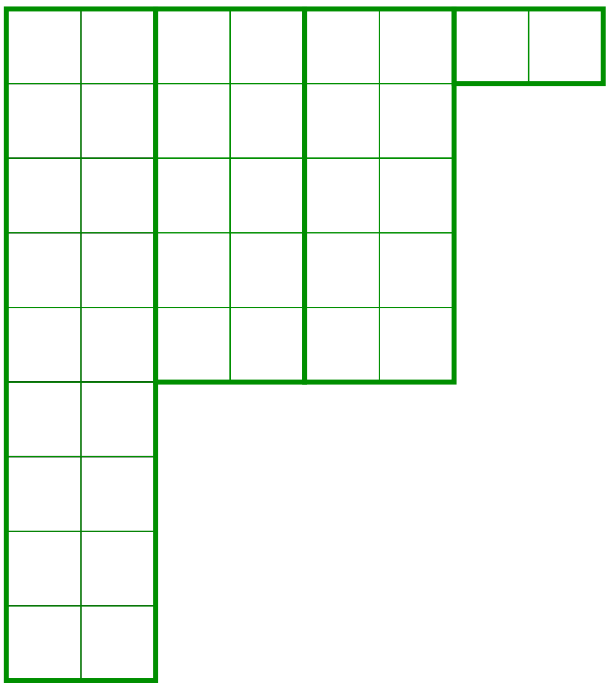
$$\sum_{\lambda \in RB} w(\lambda) = \prod_{j=1}^{\infty} \frac{(1 + a^j b^j c^j d^{j-1})(1 + a^j b^{j-1} c^{j-1} d^{j-1})}{(1 - a^j b^j c^j d^j)(1 - a^j b^{j-1} c^j d^{j-1})}$$

# Proof: Generating Function for $G$

$$G = \{\lambda \in \mathrm{Par} : \lambda' \text{ has only odd parts, each of which is repeated an even number of times}\}$$



- $\lambda \in G$

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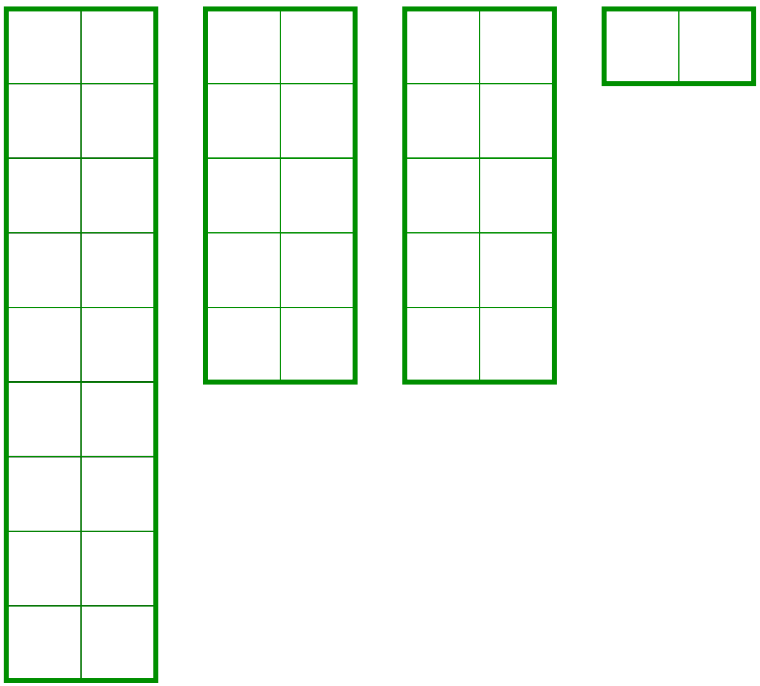

- $\lambda \in G$
- pairs of columns of odd height

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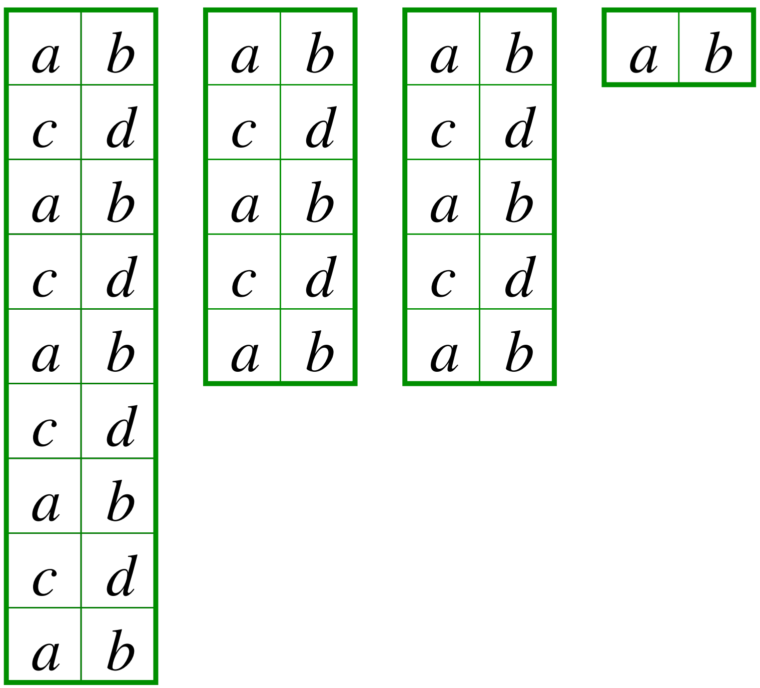


- $\lambda \in G$
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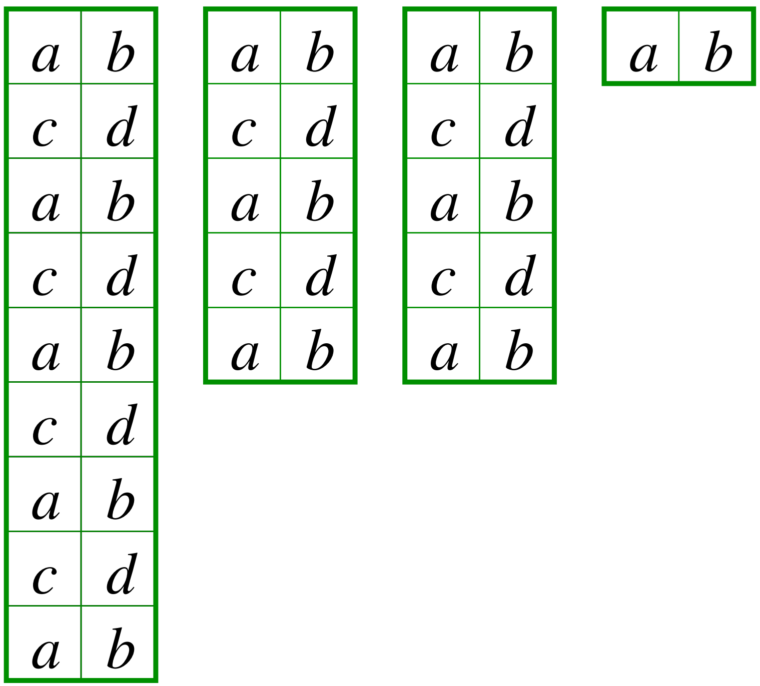


- $\lambda \in G$
- pairs of columns of odd height
- weight: $a^j b^j c^{j-1} d^{j-1}$
- may occur multiple times

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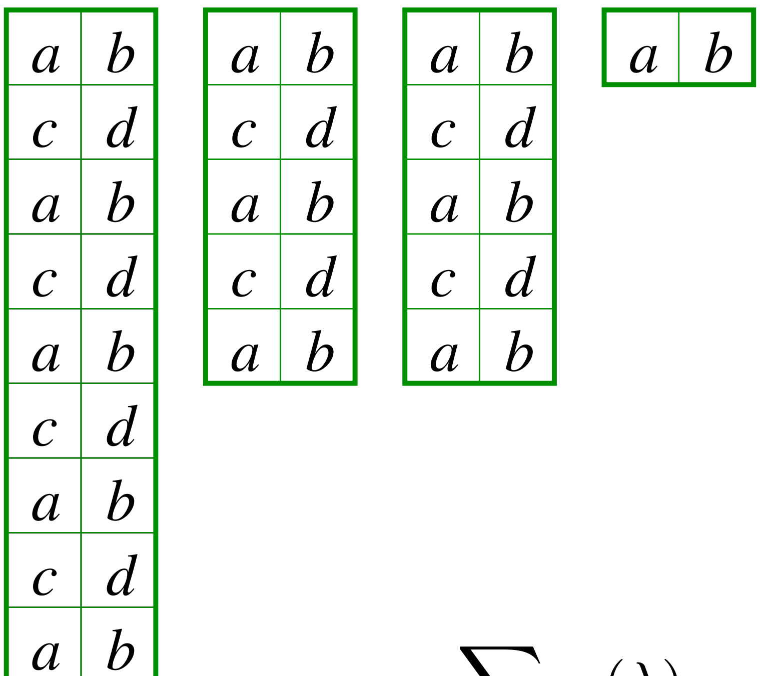


- $\lambda \in G$
- pairs of columns of odd height
- weight: $a^j b^j c^{j-1} d^{j-1}$
- may occur multiple times

$$\sum_{\lambda \in G} w(\lambda) = \prod_{j=1}^{\infty} \frac{1}{(1 - a^j b^j c^{j-1} d^{j-1})}$$

# Proof: Summary

$$\sum_{\lambda \in RB} w(\lambda) = \prod_{j=1}^{\infty} \frac{(1 + a^j b^j c^j d^{j-1})(1 + a^j b^{j-1} c^{j-1} d^{j-1})}{(1 - a^j b^j c^j d^j)(1 - a^j b^{j-1} c^j d^{j-1})}$$

$$\sum_{\lambda \in G} w(\lambda) = \prod_{j=1}^{\infty} \frac{1}{(1 - a^j b^j c^{j-1} d^{j-1})}$$

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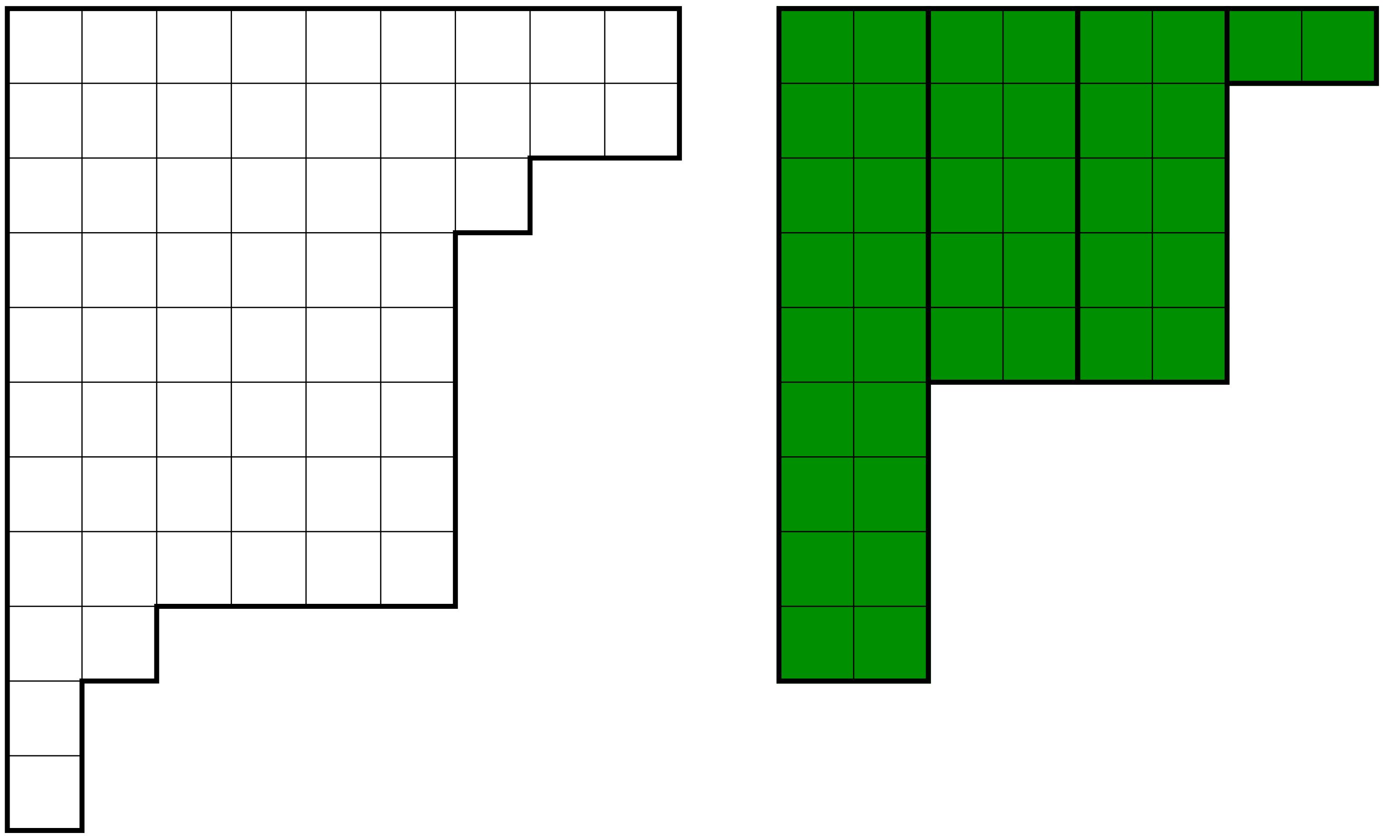
- bijection: $RB \times G \longleftrightarrow \mathrm{Par}$

## Proof: Bijection: $RB \times G \longleftrightarrow \mathrm{Par}$



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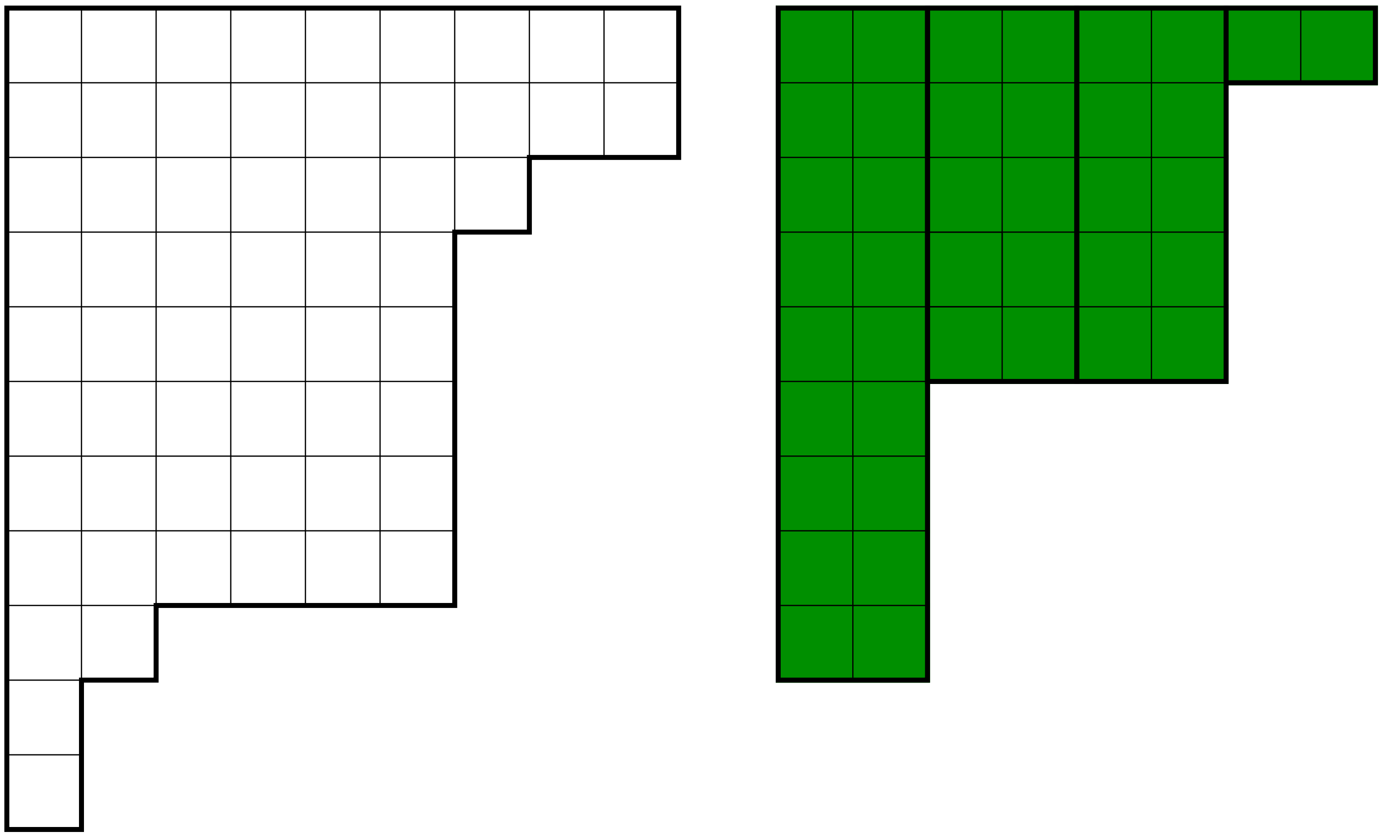

- $RB \times G \longrightarrow \mathrm{Par}$: take union of columns (sum of their parts)

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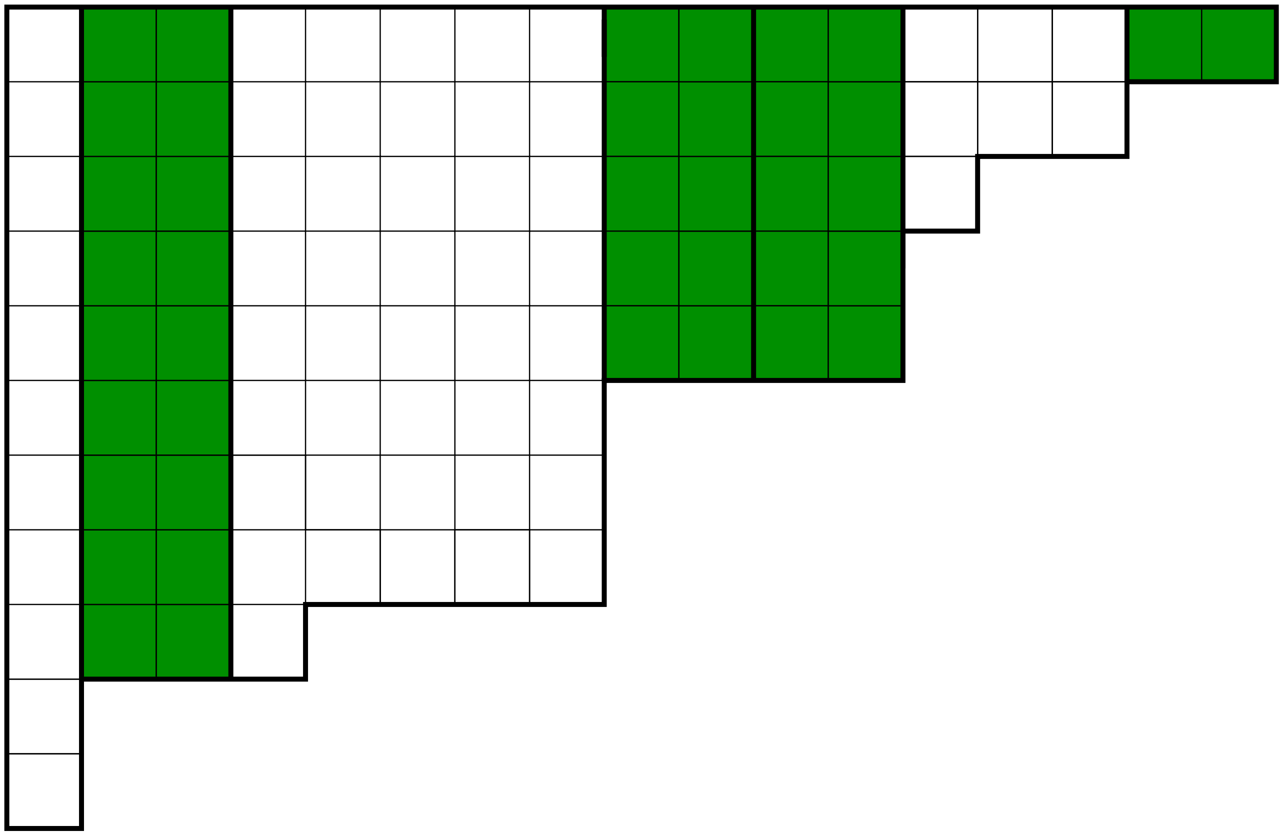

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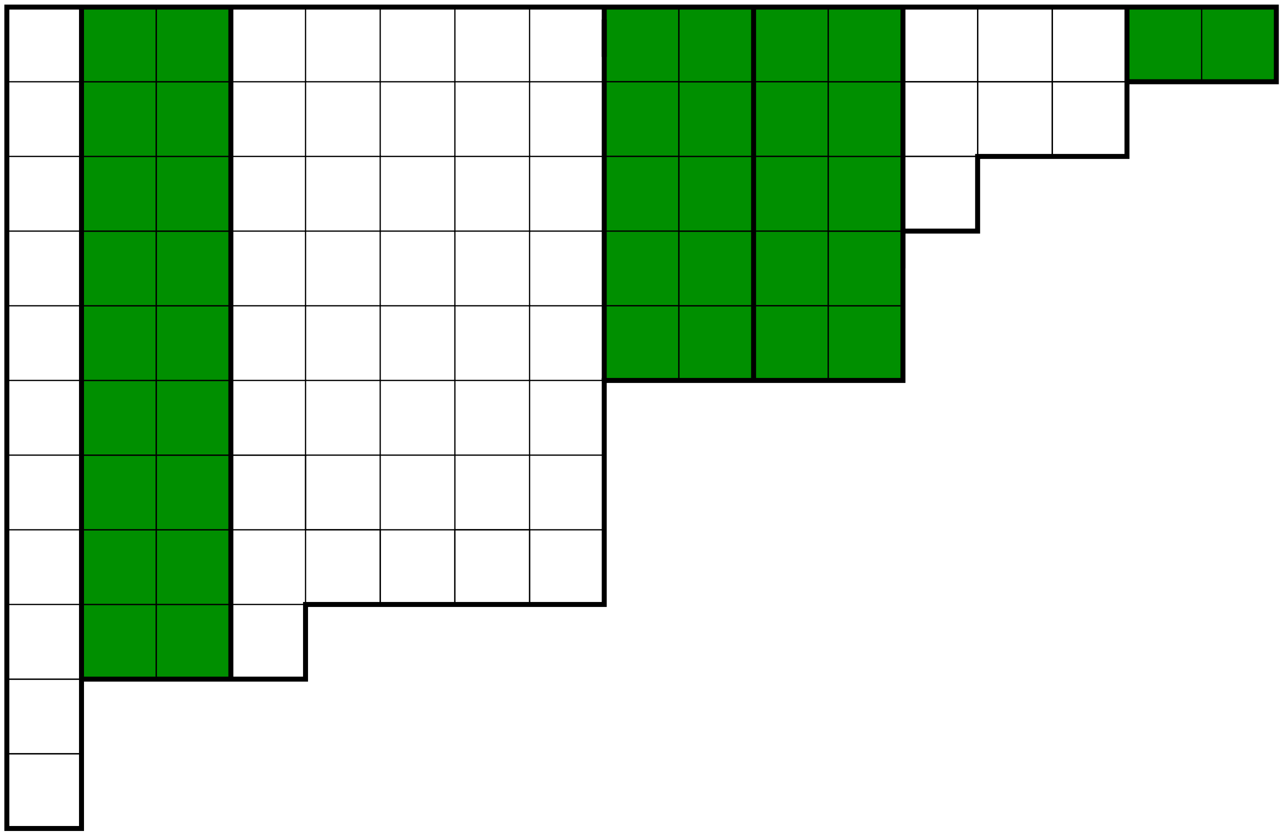
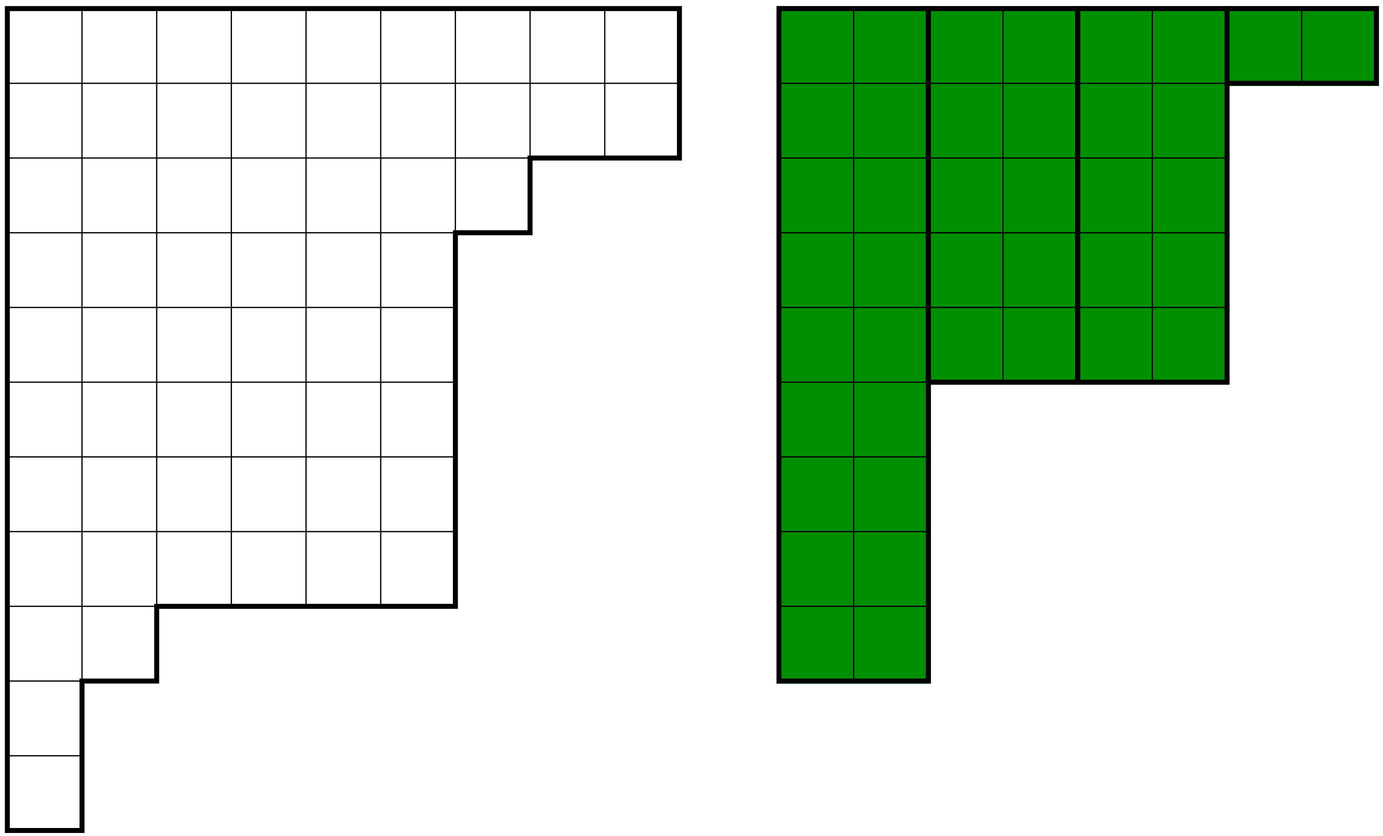


- $RB \times G \longrightarrow \mathrm{Par}$: take union of columns (sum of their parts)
- $RB \times G \longleftarrow \mathrm{Par}$: remove as many pairs of columns of odd height as possible

# Proof: Bijection: $RB \times G \longleftrightarrow \mathrm{Par}$



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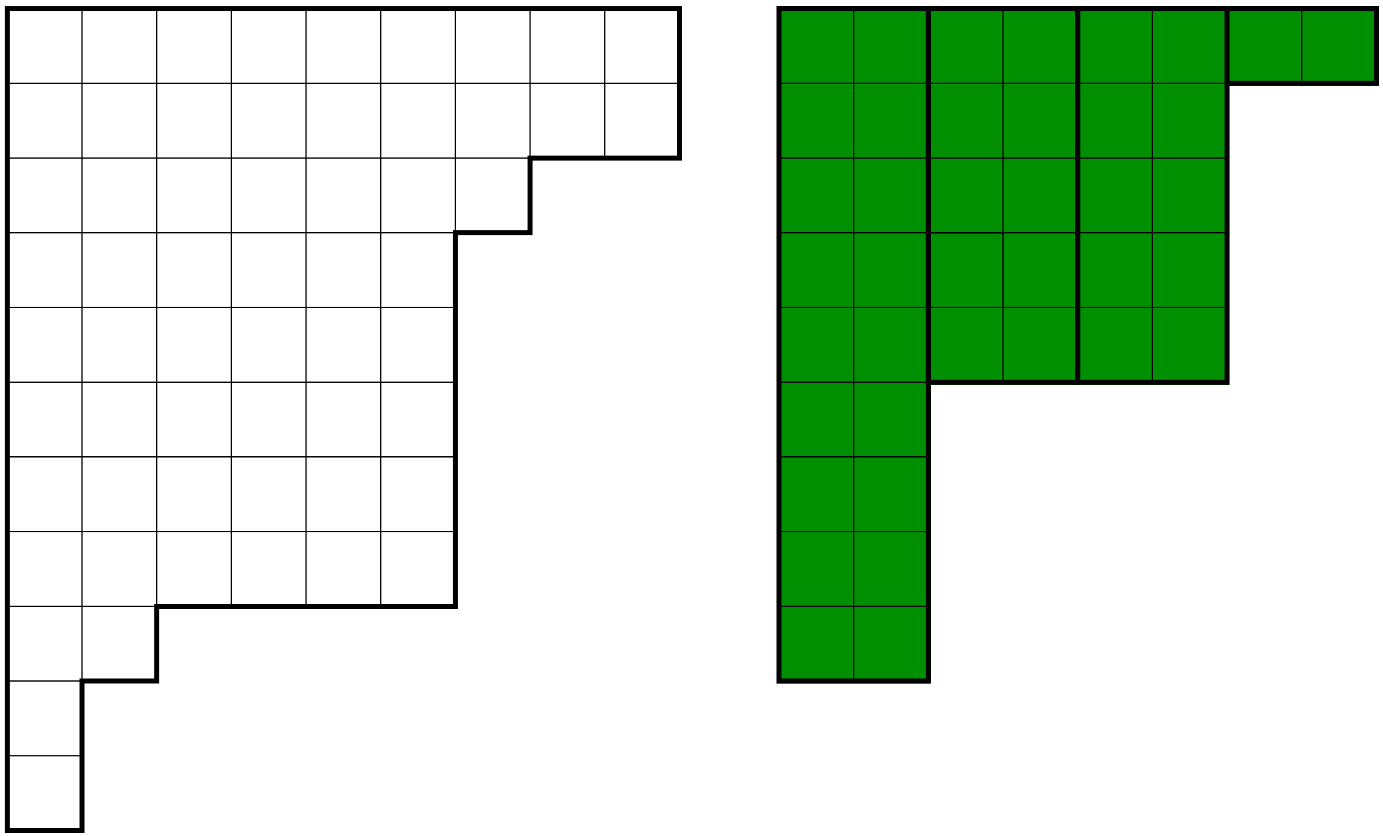
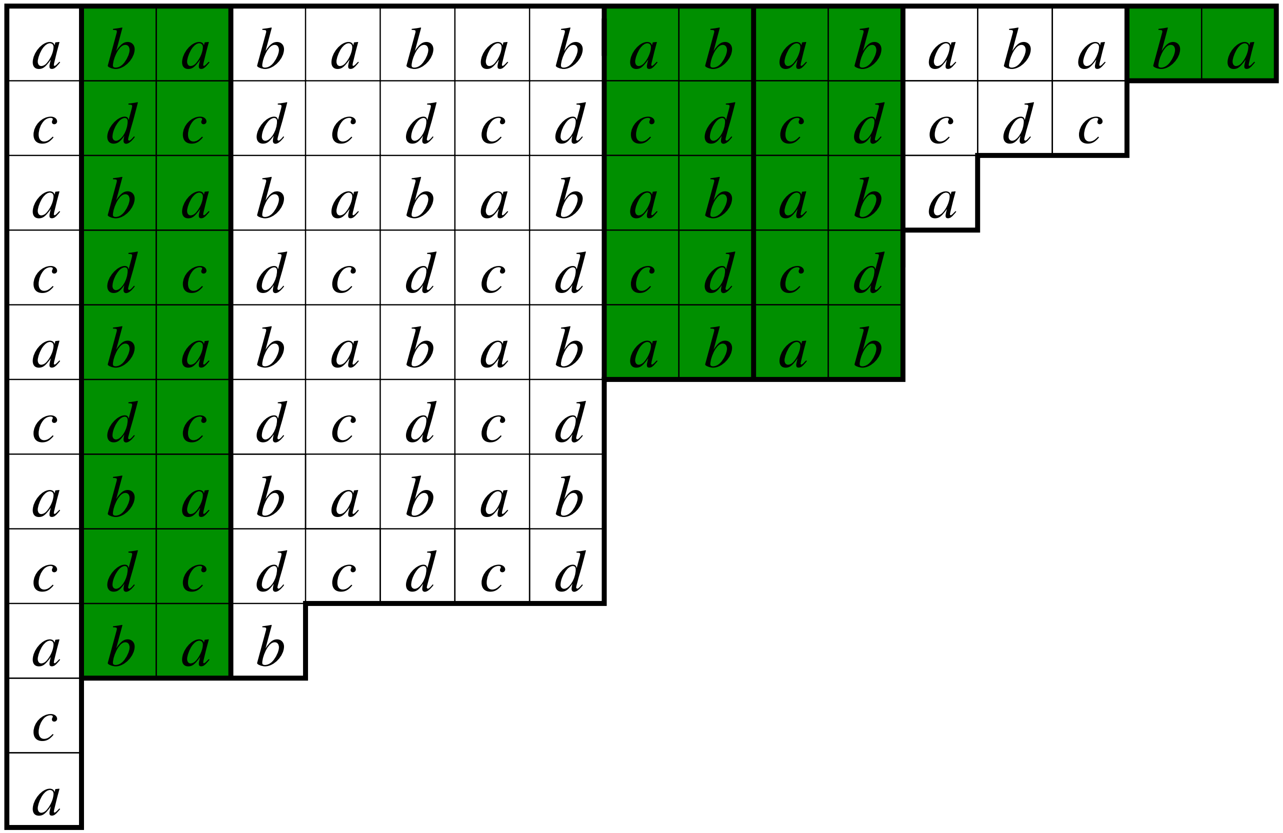


- $RB \times G \longrightarrow \mathrm{Par}$: take union of columns (sum of their parts)
- $RB \times G \longleftarrow \mathrm{Par}$: remove as many pairs of columns of odd height as possible
- gives exactly partitions such that $\lambda_{2i-1} - \lambda_{2i} \leq 1$

# Proof: Bijection: $RB \times G \longleftrightarrow \mathrm{Par}$



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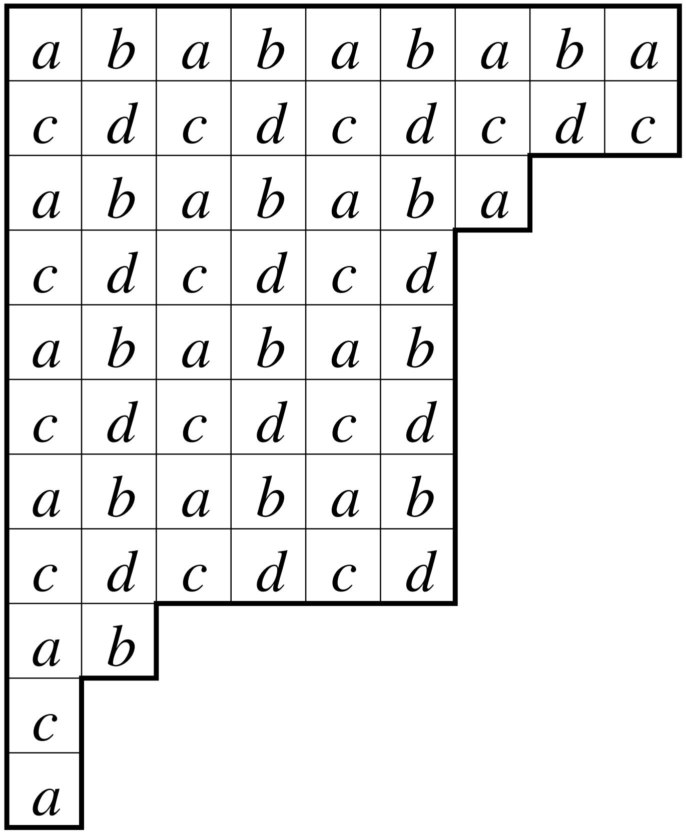


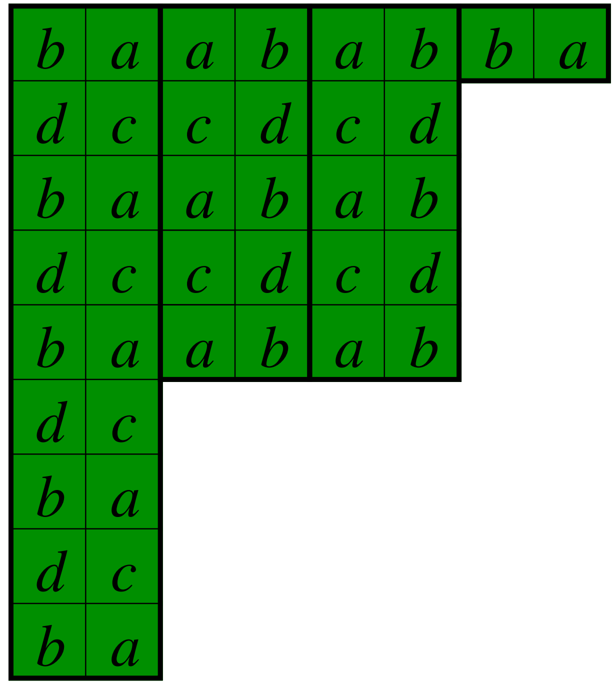

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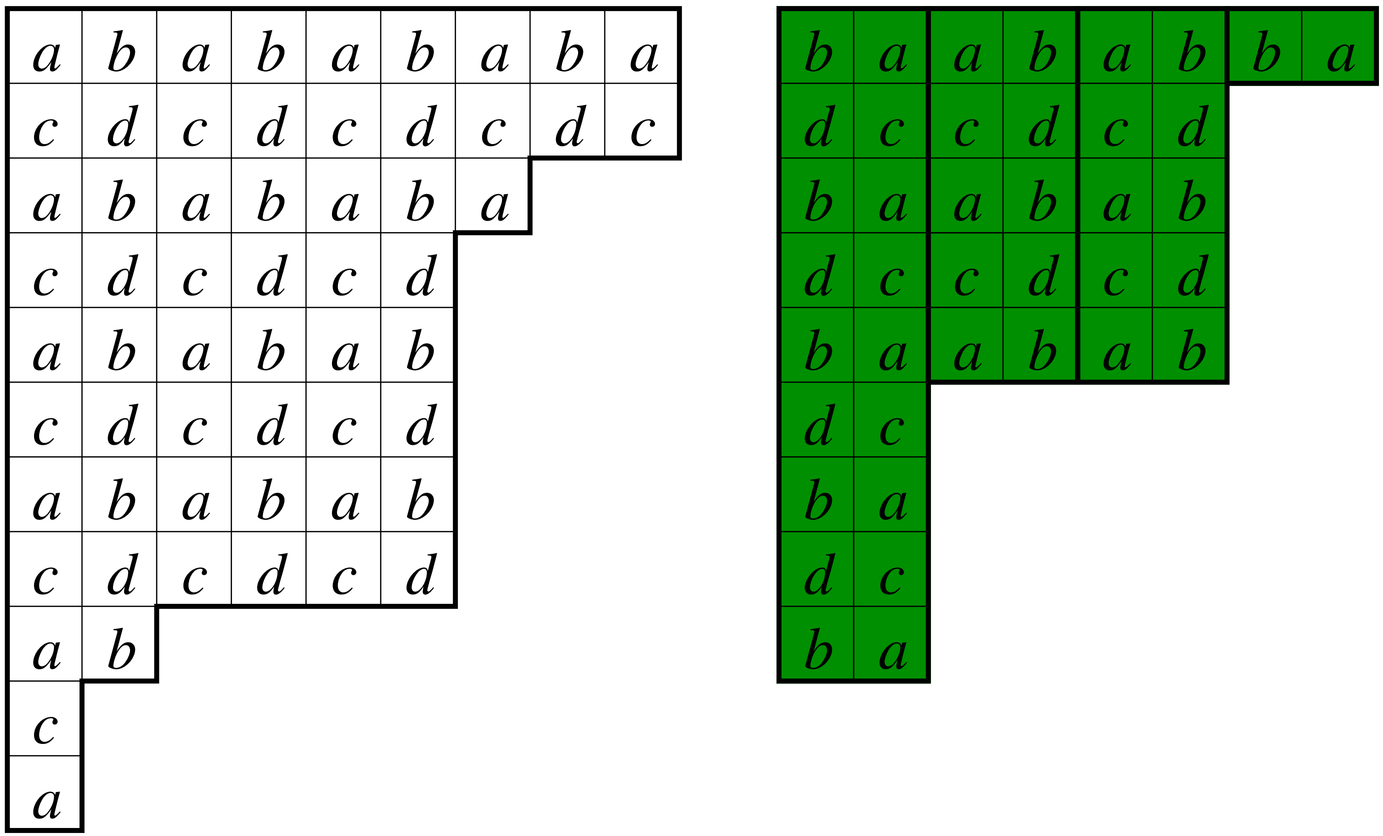
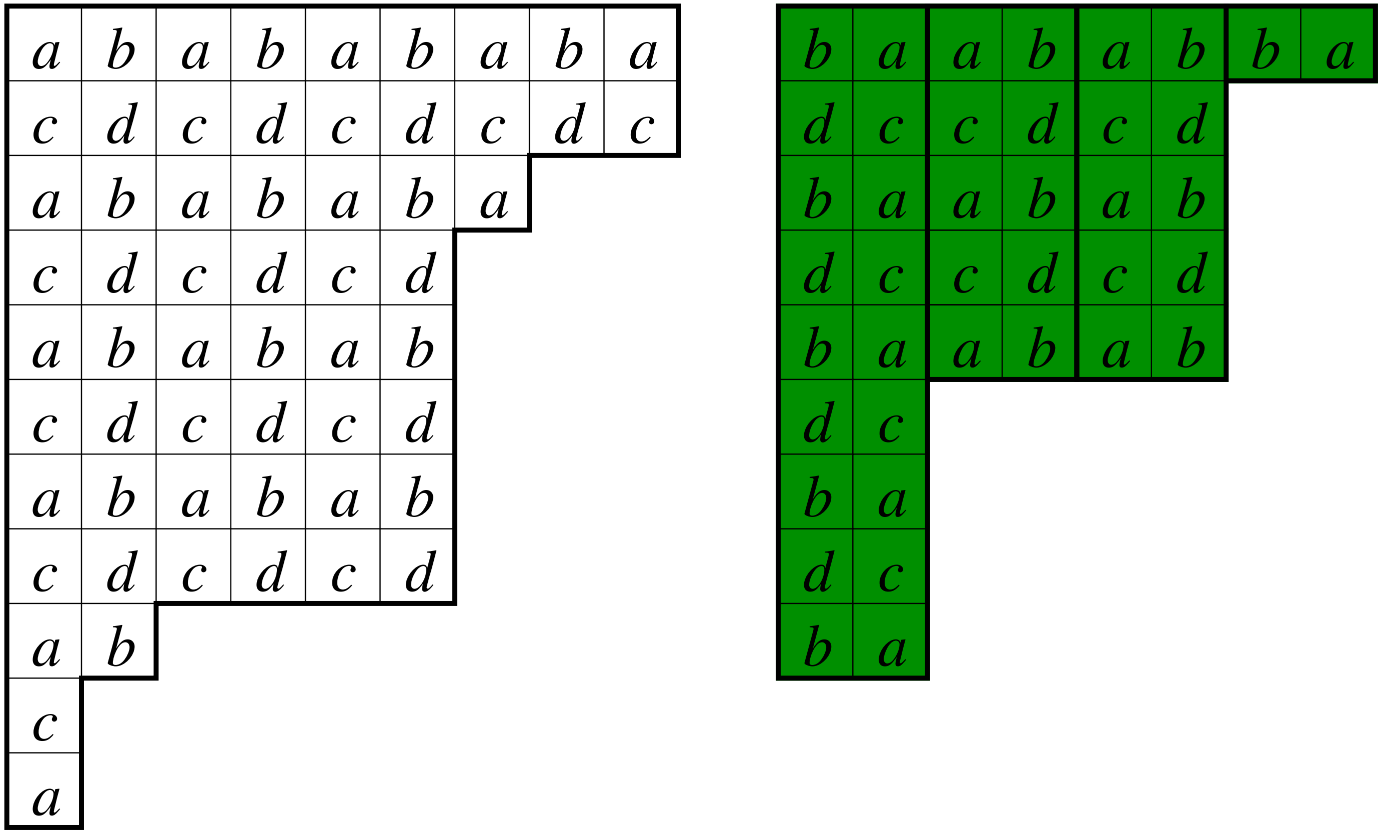



- $RB$: entries are the same

# Proof: Bijection: $\color{red}{R}\color{blue}{B}\times\color{green}{G}\longleftrightarrow \mathrm{Par}$




- $RB$: entries are the same
- $G$: pairs of columns may need to be interchanged to put labels in correct order, but overall weight is the same

## Proof: Conclusion

- bijection: $RB \times G \longleftrightarrow \mathrm{Par}$

$$\sum_{\lambda \in RB} w(\lambda) = \prod_{j=1}^{\infty} \frac{(1 + a^j b^j c^j d^{j-1})(1 + a^j b^{j-1} c^{j-1} d^{j-1})}{(1 - a^j b^j c^j d^j)(1 - a^j b^{j-1} c^j d^{j-1})}$$

$$\sum_{\lambda \in G} w(\lambda) = \prod_{j=1}^{\infty} \frac{1}{(1 - a^j b^j c^{j-1} d^{j-1})}$$

# Proof: Conclusion

- bijection: $RB \times G \longleftrightarrow \mathrm{Par}$

$$\sum_{\lambda \in \mathrm{Par}} w(\lambda) = \prod_{j=1}^{\infty} \frac{(1 + a^j b^j c^j d^{j-1})(1 + a^j b^{j-1} c^{j-1} d^{j-1})}{(1 - a^j b^j c^j d^j)(1 - a^j b^{j-1} c^j d^{j-1})(1 - a^j b^j c^{j-1} d^{j-1})}$$

Let $D$ denote the set of all partitions with distinct parts.

**Corollary 2.**

$$\sum_{\lambda \in D} w(\lambda) = \prod_{j=1}^{\infty} \frac{(1 + a^j b^{j-1} c^{j-1} d^{j-1})(1 + a^j b^j c^j d^{j-1})}{(1 - a^j b^j c^{j-1} d^{j-1})}$$

# Proof: (by pictures again)

$$\sum_{\lambda \in D} w(\lambda) = \prod_{j=1}^{\infty} \frac{(1 + a^j b^j c^j d^{j-1})(1 + a^j b^{j-1} c^{j-1} d^{j-1})}{(1 - a^j b^j c^{j-1} d^{j-1})}$$

$$\sum_{\lambda \in \mathrm{Par}} w(\lambda) = \prod_{j=1}^{\infty} \frac{(1 + a^j b^j c^j d^{j-1})(1 + a^j b^{j-1} c^{j-1} d^{j-1})}{(1 - a^j b^j c^j d^j)(1 - a^j b^{j-1} c^j d^{j-1})(1 - a^j b^j c^{j-1} d^{j-1})}$$

# Proof: (by pictures again)

$$\sum_{\lambda \in D} w(\lambda) = \prod_{j=1}^{\infty} \frac{(1 + a^j b^j c^j d^{j-1})(1 + a^j b^{j-1} c^{j-1} d^{j-1})}{(1 - a^j b^j c^{j-1} d^{j-1})}$$

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$P = \{\lambda \in \mathrm{Par} :$ every part of $\lambda$ is repeated an even number of times$\}$

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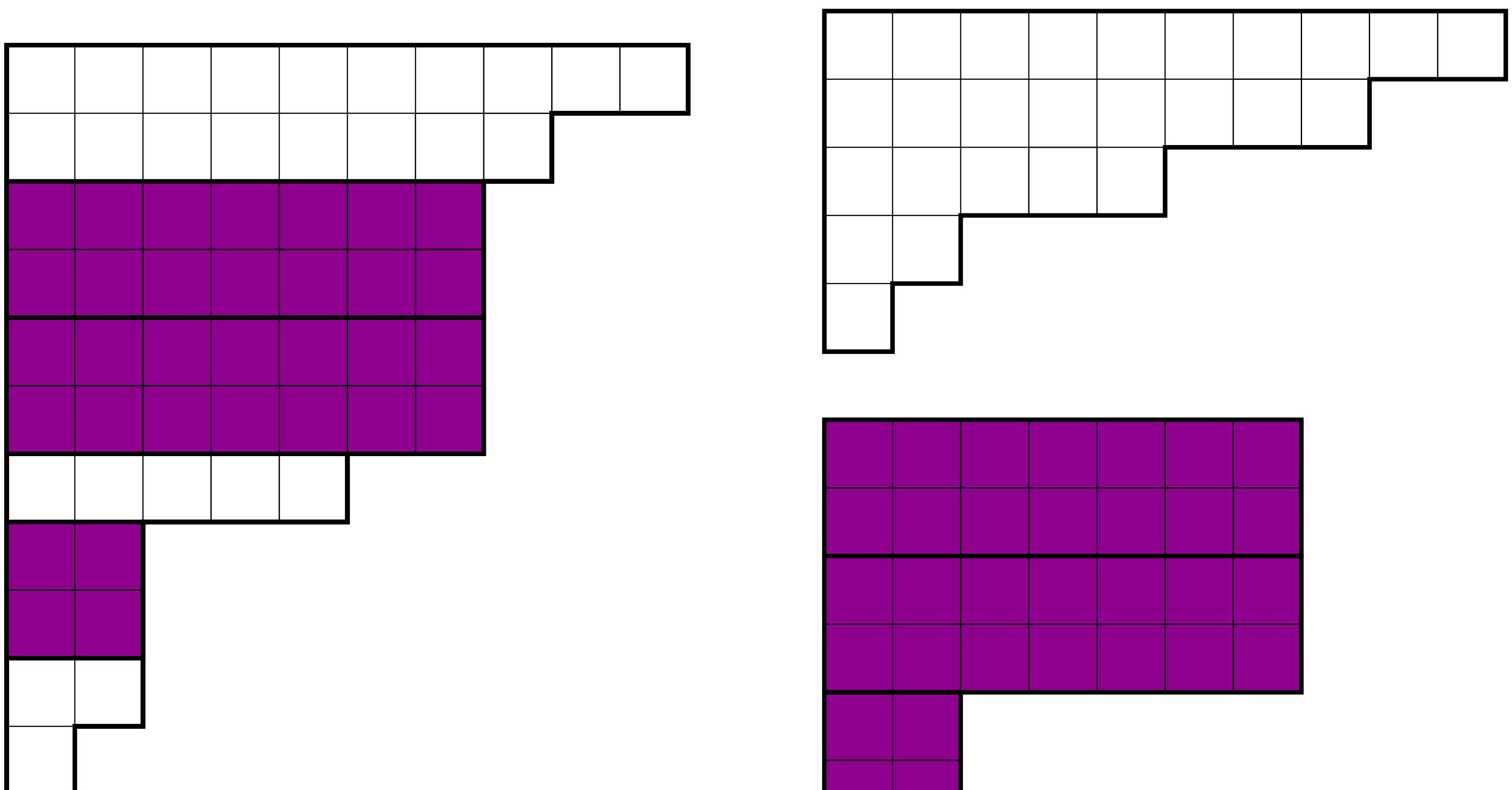
$P = \{\lambda \in \mathrm{Par} :$ every part of $\lambda$ is repeated an even number of times$\}$

- bijection: $\mathrm{Par} \longleftrightarrow D \times P$

## Proof: (by pictures again)

- bijection: $\mathrm{Par} \longleftrightarrow D \times P$



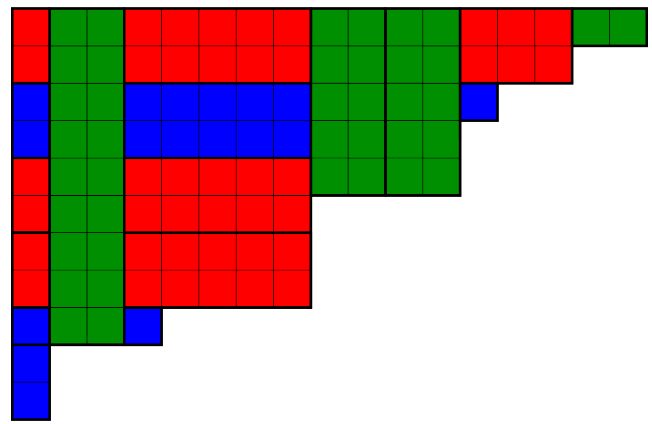
## Theorem 1.

$$\sum_{\lambda \in \mathrm{Par}} w(\lambda) = \prod_{j=1}^{\infty} \frac{(1+a^j b^j c^j d^{j-1})(1+a^j b^{j-1} c^{j-1} d^{j-1})}{(1-a^j b^j c^j d^j)(1-a^j b^{j-1} c^j d^{j-1})(1-a^j b^j c^{j-1} d^{j-1})}$$

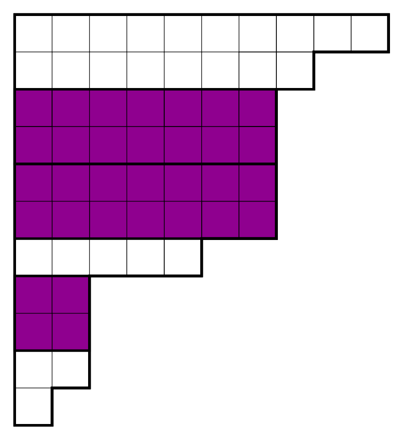
## Corollary 2.

$$\sum_{\lambda \in D} w(\lambda) = \prod_{j=1}^{\infty} \frac{(1+a^j b^{j-1} c^{j-1} d^{j-1})(1+a^j b^j c^j d^{j-1})}{(1-a^j b^j c^{j-1} d^{j-1})}$$

## Theorem 1.



## Corollary 2.



# A more general result

Let $R$ be a subset of positive integers congruent to $i (\mathrm{mod}\ k)$ and let $\rho$ be a map from $R$ to the even positive integers.
Let $\mathrm{Par}(i, k; R, \rho)$ be the set of all partitions with parts are congruent to $i (\mathrm{mod}\ k)$ such that if $r \in R$, then $r$ appears as a part less than $\rho(r)$ times.

**Theorem 3.**

$$\sum_{\lambda \in \mathrm{Par}(i,k;R,\rho)} w(\lambda) = ST$$

*where*

$$S = \prod_{j=1}^{\infty} \frac{(1 + a^{\lceil \frac{(j+1)k+i}{2} \rceil} b^{\lfloor \frac{(j+1)k+i}{2} \rfloor} c^{\lceil \frac{jk+i}{2} \rceil} d^{\lfloor \frac{jk+i}{2} \rfloor})}{(1 - (ac)^{\lceil \frac{jk+i}{2} \rceil} (bd)^{\lfloor \frac{jk+i}{2} \rfloor})(1 - (ac)^{jk} (bd)^{(j-1)k})}$$

*and*

$$T = \prod_{r \in R} (1 - (ac)^{\lceil \frac{r}{2} \rceil \frac{\rho(r)}{2}} (bd)^{\lfloor \frac{r}{2} \rfloor \frac{\rho(r)}{2}})$$