Problem Set 6 Solution

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8.1 Equation 8.10:

$$\sum_{i=1}^{n}\frac{d^2u}{d\xi_i^2}+\sum_{i=1}^{n}B_i\frac{du}{d\xi_i}+Cu=D$$

Set $D=0$ and $v=u\ exp(\sum_{i=1}^{n}\frac{B_i\xi_i}{2})$, thus $u=v\ exp(-\sum_{i=1}^{n}\frac{B_i\xi_i}{2})$

$$\frac{du}{d\xi_i}=-\frac{B_i}{2}v\ exp(-\sum_{i=1}^{n}\frac{B_i\xi_i}{2})+\frac{dv}{d\xi_i}exp(-\sum_{i=1}^{n}\frac{B_i\xi_i}{2})$$

$$\frac{d^2u}{d\xi_i^2}=-\frac{B_i}{2}(\frac{dv}{d\xi_i}exp(-\sum_{i=1}^{n}\frac{B_i\xi_i}{2})-\frac{B_i}{2}v\ exp(-\sum_{i=1}^{n}\frac{B_i\xi_i}{2}))+\frac{d^2v}{d\xi^2}exp(-\sum_{i=1}^{n}\frac{B_i\xi_i}{2})-\frac{B_i}{2}exp(-\sum_{i=1}^{n}\frac{B_i\xi_i}{2})\frac{dv}{d\xi_i}$$

Plug into equation 8.10 and divide by $exp(-\sum_{i=1}^{n}\frac{B_i\xi_i}{2})$:

$$\sum_{i=1}^{n}(-\frac{B_i}{2}\frac{dv}{d\xi_i}+\frac{B_i^2}{4}v+\frac{d^2v}{d\xi_i^2}-\frac{B_i}{2}\frac{dv}{d\xi_i})+\sum_{i=1}^{n}(-\frac{B_i^2}{2}v+B_i\frac{dv}{d\xi_i})+Cv=0$$

Thus $\sum_{i=1}^{n}\frac{d^2v}{d\xi_i^2}-\sum_{i=1}^{n}\frac{B_i^2}{4}V+Cv=0$.

Thus $\sum_{i=1}^{n}\frac{d^2v}{d\xi_i^2}+(C-\sum_{i=1}^{n}\frac{B_i^2}{4})v=0$.

Let $\lambda=C-\sum_{i=1}^{n}\frac{B_i^2}{4}$, we have $\sum_{i=1}^{n}\frac{d^2v}{d\xi_i^2}+\lambda v=0$

8.2 Equation 8.11:

$$\sum_{i=1}^{n-1}\frac{d^2u}{d\xi_i^2}-\frac{d^2u}{d\xi_n^2}+\sum_{i=1}^{n}Bi\frac{dy}{d\xi_i}+Cu=D$$

Set $D=0$ and $u=v\ exp(-\sum_{i=1}^{n}\frac{B_i\xi_i}{2}-\frac{B_n\xi_n}{2})$. Let $p=\sum_{i=1}^{n}\frac{B_i\xi_i}{2}-\frac{B_n\xi_n}{2})$, then $p'=\frac{B_i}{2}$. using chain rule, equation 8.11 becomes:

$$\sum_{i=1}^{n-1}\frac{d}{d\xi_i}(\frac{dv}{d\xi_i}e^{-p}-p've^{-p})-\frac{d}{d\xi_n}(\frac{dv}{d\xi_n}e^{-p}-p've^{-p})+\sum_{i=1}^{n}(\frac{dv}{d\xi_i}e^{p}-p've^{-p})+Cve^{-p}=0$$

Thus,

$$e^{-p}(n-1)(\sum_{i=1}^{n-1}\frac{d^2v}{d\xi_i^2}-(p')^2v-2p'\frac{dv}{d\xi_i})-e^{-p}(\frac{d^2v}{d\xi_n^2}+(p')^2v-p'\frac{dv}{d\xi_n}+p\frac{dv}{d\xi_n})+ne^{-p}(\sum_{i=1}^{n}(B_i\frac{dv}{d\xi_i}-p'v))+Cve^{-p}=0$$

Thus,

$$\sum_{i=1}^{n-1} \frac{d^2v}{d\xi_i^2} - \frac{d^2v}{d\xi_n^2} + (c/n - (\sum_{i=1}^{n-1} \frac{B_i^2}{4} - \frac{B_n^2}{4}))v = 0$$

Thus $\sum_{i=1}^{n-1} \frac{d^2v}{d\xi_i^2} - \frac{d^2v}{d\xi_n^2} + \lambda v = 0$, where $\lambda = c/n - (\sum_{i=1}^{n-1} \frac{B_i^2}{4} - B_n^2/4) = c/n - \sum_{i=1}^{n} \frac{B_i^2}{4}$.

9.1 $Pu = (\frac{du}{dx} + \frac{du}{dy})^2 - u^2$, $u_1 = e^x$, $u_2 = e^{-y}$

$\frac{du_1}{dx} = e^x$, $\frac{du_1}{dy} = 0$, $\frac{du_2}{d_x} = 0, \frac{du_2}{dy} = -e^{-y}$

$Pu_1 = (e^x)^2 - (e^x)^2 = 0;$

$Pu_2 = (-e^{-y})^2 - (e^{-y})^2 = 0$

Thus, u_1 and u_2 are solutions to $Pu = 0$.

$u(x, y) = e^x + e^{-y}$,

$\frac{du}{dx} = e^x$, $\frac{du}{dy} = -e^{-y}$.

$Pu = (e^x - e^{-x})^2 - (e^x + e^{-y})^2 = -4e^{x-y} \neq 0$. Thus, $u(x, y) = e^x + e^{-y}$ is not a solution.

9.2 (a) Is $u = c_1u_1 + c_2u_2$ a solution of $Pu = c1f1 + c2f2$?

$P(c_1u_1 + c_2u_2) = P(c_1u_1) + P(c_2u_2) = c_1Pu_1 + c_2Pu_2 = c_1f_1 + c_2f_2$.

Thus, $u = c_1u_1 + c_2u_2$ a solution.

(b) $Pu(x, \lambda) = f(x, \lambda)$

Is $u(x) = \int_I g(\lambda)u(x, \lambda)d\lambda$ a solution of $Pu(x) = f(x)$, where $f(x) = \int_I g(\lambda)f(x, \lambda)d\lambda$?

$Pu(x) = P\int_I g(\lambda)u(x, \lambda)d\lambda{=}\int_I g(\lambda)f(x, \lambda)d\lambda = f(x)$

Thus, $u(x) = \int_I g(\lambda)u(x, \lambda)d\lambda$ is a solution.

1.3 $\int_\Omega u\nabla^2 w dv = \int_S u\frac{dw}{dn}d\delta - \int_\Omega (\nabla u)(\nabla u)dv$

set $w = u$:

$$\int_\Omega u\nabla^2 u dv = \int_S u\frac{du}{dn}d\delta - \int_\Omega (\nabla u)\nabla u)dv$$

So, $\int_\Omega (\nabla u)(\nabla u)dv = 0$.Thus,$\nabla u = (\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}) = 0$, and $\frac{du}{dx} = \frac{du}{dy} = \frac{du}{dz} = 0$.

So u has to be constant in $\overline{\Omega}$ because the partial derivatives are all 0.

5.2 (a) Define the smooth surface $S \subseteq R^2$ by $S = \{(x, 0) : x \in R\}$. Then we have the cauchy problem,

$$D_x^2u + D_y^2u = 0 = f((x, y))$$

$$u = 0 = \phi_0((x, y)) \text{ on } S$$

$\frac{du}{dn} = (D_xu, D_yu)\dot{(}0, 1) = D_yu = \frac{1}{n}\sin(nx) = \phi_1((x, y))$ on S.

Recall from page 142 that the PDE has no characteristic curves, since Lplace's equation is elliptic. Thus, since the coefficients of PDE and the functions $f_1\phi_0$ and ϕ_1 are all analytic at any point of S, the general Cauchy-Kovalevsky Theorem on pages 132-133 guarantees that the

Cauchy problem has a unique solution in the class of anlytic functions.

We now verify that $u(x,y) = \frac{1}{n^2}\sinh(ny)\sin(nx)$, is a solution to the Cauchy problem and the therefore the unique solution:
$\frac{d^2u}{dy^2} + \frac{d^2}{dx^2} = \frac{1}{n^2}(n^2 sinh(ny)\sin(nx) - n^2 sinh(ny)\sin(nx)) = 0$
$u|_s = u(x,0) = \frac{1}{n^2} sinh(n0)\sin(nx) = 0$
$\frac{du}{dy}|_s = \frac{du}{dy}(x,0) = \frac{1}{n} cosh(n0)\sin(nx) = \frac{1}{n} sin(nx)$

Hence $u(x,y)$ is indeed the unique solution to the Cauchy problem.

(b) $sinh(x)$ is a rapidly growing function as $s \to \infty$, and $\sin(x)$ is alwaya between -1 and 1. So for the inital data, $|U_y(x,0)| = |\frac{1}{n}\sin(nx)|$ is arbitrarily small when n is large because $1/n$ becomes small while $|\sin(nx)|$ stays between -1 and 1.

On the other hand, the solution takes arbitrarily large values because eventhough $1/n^2$ becomes small, $sinh(ny)$ becomes large very fast (even when $|y|$ is relatively small).

(c) I. $\frac{d^2u_1}{dy^2} + \frac{d^2u_1}{dx^2} = 0$, $u_1(x,0) = f(x)$, $U_{(1)y}(x,0) = g(x)$
II. $\frac{d^2u_2}{dy^2} + \frac{d^2u_2}{dx^2} = 0$, $u_2(x,0) = f(x)$, $u_{(2)y}(x,0) = g(x) + \frac{1}{n}\sin(nx)$
Subtract I from II:
(1) $(\frac{d^2u_2}{dy^2} + \frac{d^2u_2}{dx^2}) - (\frac{d^2u_1}{dy^2} + \frac{d^2u_1}{dx^2}) = \frac{d^2(u_2-u_1)}{2y^2} + \frac{d^2(u_2-u_1)}{dx^2} = 0$
(2) $u_2(x,0) - u_1(x,0) = f(x) - f(x) = 0$
(3) $U_{(2)y}(x,0) - u_{(1)y}(x,0) = g(x) + \frac{1}{n}\sin(nx) - g(x) = \frac{1}{n}\sin(nx)$
So, $u_2(x,y) - u_1(x,y) = \frac{1}{n^2} sinh(ny) sin(nx)$.

(d) Consider two instances of the Cauchy problems for laplace's equation:
$\frac{d^2u}{dy^2} + \frac{d^2u}{dx^2} = 0$
$u(x,0) = f(x)$
$u_y(x,0) = g(x)$

$\frac{d^2u}{dy^2} + \frac{d^2u}{dx^2} = 0$
$u(x,0) = f(x)$
$u_y(x,0) = g(x) + \frac{1}{n}\sin(nx) = g_n(x)$

with solutions $u(x,y)$ and $u_n(x,y)$ respectively, (for any value of $n > 0$). By analyzing the differences in the initial data g and g_n and in the solutions u and u_n for the above problems, we wish to show that the solution to the Cauchy problem for Laplace's equation does not depend continuously on the intial data. Thus, we wish to prove the negation of the statement.

$\forall\epsilon > 0$, $\exists\delta > 0$, s.t., $\forall n$: $max_{(x,y)\in S}|g - g_n| < S \Rightarrow \max_{(x,y)\in R^2}|u - u_n| < \epsilon$, where S is the inital curve $\{(x,0) : x\} \subseteq R^2$.

Consequently, we need to show that, $\exists\epsilon > 0$, s.t. $\forall\delta > 0$, $\exists n$, s.t. : $\max_{(x,y)\in S}|g - g_n| < \delta$ and $\max_{(x,y)\in S}|u - u_n| \geq \epsilon$.

Using the results from part (c), this statement is equivalent to the statement $\exists\epsilon > 0$, s.t. $\forall\delta > 0$, $\exists n$, s.t. : $\max_{(x,y)\in S}|\frac{1}{n}\sin(nx| < \delta$ and $\max_{(x,y)\in R^2}|\frac{1}{n^2} sinh(ny)\sin(nx)| \geq \epsilon$.

To proceed, choose any $\epsilon > 0$. Take any $\delta > 0$. now recall that we have shown that $\frac{1}{n}\sin(nx)$ takes everywhere arbitrarily small values as $n \to \infty$ and that there exists a point $(x, y) \in R^2$ such that $\frac{1}{n^2} sinh(ny)\sin(nx)$ takes arbitrarily large values as $n \to \infty$. Hence, we can find n sufficiently large such that,

$$\max_{(x,y)\in S} |\frac{1}{n}\sin(nx)| < \delta$$

and

$$\max_{(x,y)\in R^2} |\frac{1}{n^2}\sinh(ny) sin(nx)| \geq \epsilon$$

This proves the desired claim. We have shown that a small perturbation in the initial data can lead to a large pertubation in the solution to the Cauchy problem; the Cauchy problem for Laplace's equation in R^2 therefore does not depend continuously on the initial data.