Problem 1

(a) $f(x) = logx$, $x_0 = 1$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$
$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$
$$= \frac{f^{(0)}(1)(x-1)^0}{0!} + \frac{f^{1}(1)(x-1)}{1!} + \frac{f^{(2)}(1)(x-1)^2}{2!} + \frac{f^{(3)}(1)(x-1)^3}{3!} + \frac{f^{(4)}(1)(x-1)^4}{4!} + \frac{f^{(5)}(1)(x-1)^5}{5!}$$
$$= 0 + (x-1) - \frac{(x-1)^2}{2} + \frac{2(x-1)^3}{6} - \frac{4(x-1)^4}{24} + \frac{24(x-1)^5}{120} - \cdots, \ 0 < x < 2$$

(b) $f(x) = \frac{1}{1-x^2}$, $x_0 = 0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)(x-x_0)^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$$
$$= \sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^8 \cdots, \ -1 < x < 1$$

(c) $f(x) = cos(x)$, $x_0 = \frac{\pi}{4}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)(x-x_0)^n}{n!} = \sum_{n=0}^{\infty} \frac{f^{(n)}(\frac{\pi}{4})(x-\frac{\pi}{4})}{n!}$$

$$f(x) = cosx, \ f(x_0) = cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$
$$f'(x) = -sinx, \ f'(x_0) = -sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$$
$$f''(x) = -cosx, \ f''(x_0) = -cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$$
$$f^{(3)}(x) = sinx, \ f^{(3)}(x_0) = sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$
$$f^{(4)}(x) = cosx, \ f^{(4)}(x_0) = cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

thus $cosx = \frac{\sqrt{2}}{2}(1 - (x - \frac{\pi}{4}) - \frac{(x-\frac{\pi}{4})^2}{2} + \frac{(x-\frac{\pi}{4})^3}{6} + \frac{(x-\frac{\pi}{4})^4}{24} ...)$, $x \in R$

(d) $f(x) = sin2x$, $x_0 = 0$, $f(x_0) = 0$

$$f'(x) = 2cos2x, \ f'(x_0) = 2$$
$$f''(x) = -4sin2x, \ f''(x_0) = 0$$
$$f^{(3)}(x) = -8cos2x, \ f^{(3)}(x_0) = -8$$
$$f^{(4)}(x) = 16sin2x, \ f^{(4)}(x_0) = 0$$
$$f^{(5)}(x) = 32cos2x, \ f^{(5)}(x_0) = 32$$
$$f^{(6)}(x) = -64sin2x, \ f^{(6)}(x_0) = 0$$
$$f^{(7)}(x) = -128cos2x, \ f^{(7)}(x_0) = -128$$
$$f^{(8)}(x) = 256sin2x, \ f^{(8)}(x_0) = 0$$
$$f^{(9)}(x) = 512cos2x, \ f^{(9)}(x_0) = 512$$

$$f(x) = sin2x = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$
$$= \frac{f'(x_0)x}{1!} + \frac{f^{(3)}(x_0)x^3}{3!} + \frac{f^{(5)}(x_0)x^5}{5!} + \frac{f^{(7)}(x_0)x^7}{7!} + \frac{f^{(9)}(x_0)x^9}{9!} \cdots$$
$$= 2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \frac{128x^7}{5040} + \frac{512x^9}{342880} \cdots$$
$$= 2x - \frac{4x^3}{3} + \frac{4x^5}{15} - \frac{8x^7}{315} + \frac{4x^9}{2835} ..., \ x \in R$$

Problem 2

(c) $u_x = (sinu)u_x$, $u(0, x) = \frac{\pi}{6} + x$

first $\phi(x) = \frac{\pi}{6} + 4$, implies it is analytic

Obviously, $F(t,x,u,p) = (sinu)p$ is also analytic

Therefore, $u(0,x) = \frac{\pi}{6} + x$
$u_x(0,x) = 1$
$u_{xx}(0,x) = 0$
$u_t(0,x) = (sinu)u_x = sin(\frac{\pi}{6} + x)u_x$
$u_{tx}(0,x) = cos(\frac{\pi}{6} + x)u_x + u_{xx}sin(\frac{\pi}{6} + x)$
$u_{tt} = sin(\frac{\pi}{6} + x)u_{xt}$
$\quad = (sin\frac{\pi}{6} + x)(cos(\frac{\pi}{6} + x)u_x + u_{xx}sin(\frac{\pi}{6} + x))$

$u(0,0) = \frac{\pi}{6}$
$u_x(0,0) = 1$
$u_{xx}(0,0) = 0$
$u_t(0,0) = \frac{1}{2}$
$u_{tx}(0,0) = \frac{\sqrt{3}}{2}$
$u_{tt}(0,0) = \frac{\sqrt{3}}{2}$

Thus $u(t,x) = \frac{\pi}{6} + x + \frac{1}{2}t + \frac{\sqrt{3}}{2}tx + \frac{\sqrt{3}}{4}t^2...$

(d) $u_t = e^{tx}u_x$, $u(0,x) = 1 - x + x^2$, $u(t,x) = 1 - t - x + t^2 + 2tx + x^2$

$\phi(x) = 1 - x + x^2$ is analytic
$F(x,t,u,p) = e^{tx}p$ is also analytic.

$u(0,x) = 1 - x + x^2$
$u_x(0,x) = -1 + 2x$
$u_{xx}(0,x) = 2$
$u_t(0,x) = e^{tx}(-1 + 2x) = -e^{tx} + 2xe^{tx}$
$u_{tt}(0,x) = xe^{tx}u_x + u_{tx}e^{tx}$
$u_{tx}(0,x) = -te^{tx} + 2txe^{tx} + 2e^{tx}$

$u(0,x) = 1$
$u_x(0,0) = -1$
$u_{xx}(0,0) = 2$
$u_t(0,0) = -1$
$u_{tt}(0,0) = 2$
$u_{tx}(0,0) = 2$

$u(t,x) = 1 - x - t + 2tx + \frac{2}{2!}x^2 + \frac{2}{2!}t^2 + ... = 1 - x - t + 2t + x^2 + t^2...$

(e) $u_t = u_{x1}u_{x2}$, $u(0,x_1,x_2) = x_1 + x_2 - 2x_1^2$

Obviously, $\phi(x) = x_1 + x_2 + 2x_1^2$ is analytic, and $F(t,x,u,p,q) = pq$ is analytic.

$u(0,x_1,x_2) = x_1 + x_2 - 2x_1^2$
$u_{x1}(0,x_1,x_2) = 1 - 4x_1$
$u_{x2}(0,x_1,x_2) = 1$
$u_{x1x1}(0,x_1,x_2) = -4$
$u_{x2x2}(0,x_1,x_2) = 0$

Problem 2 continued on next page. . .

$u_{x1x2}(0, x_1, x_2) = 0$
$u_t(0, x_1, x_2) = 1 - 4x_1$
$u_{tx1}(0, x_1, x_2) = -4$
$u_{tx2}(0, x_1, x_2) = 0$
$u_{tt}(0, x_1, x_2) = -4$

$u(0, 0, 0) = 0$
$u_{x1}(0, 0, 0) = 1$
$u_{x2}(0, 0, 0) = 1$
$u_{x1x1}(0, 0, 0) = -4$
$u_{x2x2}(0, 0, 0) = 0$
$u_{x1x2}(0, 0, 0) = 0$
$u_t(0, 0, 0) = 1$
$u_{tx1}(0, 0, 0) = -4$
$u_{tx2}(0, 0, 0) = 0$
$u_{tt}(0, 0, 0) = -4$

Thus, $u(x, t) = x_1 + x_2 - \frac{4x^2}{2!} + t - 4tx_1 - \frac{4t^2}{2!} + ...$
$= x_1 + x_2 + t - 2x_1^2 - 4tx_1 - 2t^2 + ...$

Problem 3

$u_t = sinu_x$, $u(0, x) = \frac{\pi}{4}$

$\phi(x) = \frac{\pi}{4}x$ is analytic on $x \in R$
Obviously, $F(x, t, u, p) = sinp$ is also anlytic

$u(0, x) = \frac{\pi}{4}x$
$u_x(0, x) = \frac{\pi}{4}$
$u_{xx}(0, x) = 0$
$u_t(0, x)sinu_x = sin(\frac{\pi}{4})$
$u_{tx} = (cosu_x) \cdot u_{xx}$
$u_{tt} = (cosu_x)u_{xt}$

$u(0, 0) = 0$
$u_x(0, 0) = \frac{\pi}{4}$
$u_{xx}(0, 0) = 0$
$u_t(0, 0) = \frac{\sqrt{2}}{2}$
$u_{tx}(0, 0) = 0$
$u_{tt}(0, 0) = 0$

Thus, $u(t, x) = \frac{\pi}{4}x + \frac{\sqrt{2}}{2}t...$

Problem 4

(a) $y^3u_{xx} + u_{yy} = 0$, $y^3D_x^2 - Dy^2 = 0$, $\xi_x = \pm\sqrt{-y^3}\xi_y$

-when $y > 0$, there are no characteristic directions and no characteristic curves,

-when $y \leq 0$, characteristic directions at (x, y) are given by $(1, \pm\sqrt{-y})$, $y^3 D_x^2 + D_y^2 = 0$, $y^3 D_x^2 = -D_y^2$

Therefore, $dx = \pm\sqrt{-y^3}dy$
$dx = \pm(-y)^{3/2}dy$
$x = \pm\frac{2}{5}(-y)^{5/2} + c$ where $y \leq 0$

(b) $u_x + 2xyu_y + e^x u = cos(x + y)$
i.e. $D_x + 2xyDy + e^x u = cos(x + y)$
we have $\xi_x + 2xy\xi_y = 0$, the characteristic direction is $(-2xy, 1)$
Therefore, the characteristic curve is $y = ce^{-x^2}$.