Problem 1

(b) $\overline{V} = (x, -y, 0)$ integral surface of $\overline{V}$ that contains C: $x = t$, $y = t$, $z = t^2$
First let's check the tangent to C: $(1, 1, 2t) \neq (t, -t, 0)$ for all t
the first integrals are $xy = u_1$, $z = u_2$
Therefore $u_1 = xy = t \cdot t = t^2$, $u_2 = z = t^2$
thus $u_1 - u_2 = 0$ for all t
Substitute $u_1 = xy$ and $u_2 = z$, we have $xy - z = 0$

(d) $\overline{V} = (y_1 - x_1, 2xyz)$, $C : x = t, y = t, z = t^2$
First let's check the tangent to $C : (1, 1, 2t) \neq (t, -t, 2t^4)$
the first integrals are
$\frac{dx}{y} = \frac{dy}{-x} \Rightarrow -xdx = ydy \Rightarrow -\frac{x^2}{2} = \frac{y^2}{2} + const$
$\Rightarrow x^2 + y^2 = u_1 = c_1$
$\frac{dx}{y} = \frac{dz}{2xyz} \Rightarrow 2xdx = \frac{1}{z}dz \Rightarrow x^2 = lnz + const$
$\Rightarrow u_2 = x^2 - lnz = c_2$
Therefore, $u_1 = x^2 + y^2 = 2t^2 \Rightarrow t^2 = \frac{u_1}{2}$
$u_2 = x^2 - lnz = t^2 - ln(t^2)$
$u_2 = \frac{u_1}{2} - ln\frac{u1}{2}$
$\Rightarrow x^2 - lnz = \frac{x^2+y^2}{2} - ln(\frac{x^2+y^2}{2})$

Problem 2

$\Leftarrow\Rightarrow$ First let's check if C is tangent to V

(a) (i) $V = (1, 1, z) = (1, 1, e^t)$
$C : (x, y, z) = (t, 1 + t, e^t)$, $o < t < \infty$
$\Rightarrow \frac{\delta C}{\delta t} = (1, 1, e^t)$
therefore $V = \frac{\delta C}{\delta t}$

(ii) $\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{z}$
$\int dx = \int dy \Rightarrow x - y = u_1$
$\int dx = \int \frac{dz}{z} \Rightarrow x - lnz = u_2$
In this curve, $u_1 = t - (1 + t) = -1$
$u_2 = t - lne^t = t - t = 0$
Therefore, find F s.t. $F(u_1, u_2) = F(-1, 0) = 0$

(b) (i) $V = (xz, yz, -xy)$
$C : (t, -t, t) \Rightarrow \frac{\delta C}{\delta t} = (1, -1, 1)$
$V = (t^2, -t^2, t^2) = t^2(1, -1, 1)$
$\Rightarrow V = const\frac{\delta c}{\delta t}$

(ii) $\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$
$lnx = lny + const \Rightarrow \frac{x}{y} = u_1 = c_1$
$ydx = xdy = -zdz$
$\Rightarrow c_1ydy = -zdz \Rightarrow c_1 \cdot \frac{y^2}{2} = -\frac{z^2}{2} + const$
$ln - C$, $u_1 = \frac{x}{y} = \frac{t}{-t} = -1$, $u_1 = -1$
$u_2 = c_1y^2 + z^2 = c_1t^2 + t^2$
suppose $c_1 = -1$ then $u_2 = 0$

Problem 2 continued on next page. . .

thus, find F s.t. $F(u_1, u_2) = F(-1, 0) = 0$

Problem 3

(a) $\overline{V} = (1, 1, 1)$

(1) $\overline{\nabla} \cdot \overline{V} = 0$ (solenoidal $\rightarrow$ use THM5.1)

(2) $\frac{dx}{1} = \frac{dy}{1} \Rightarrow x + y = u_1$

likewise $y - z = u_2$

$\overline{\nabla} u_1 x \overline{\nabla} u_2 = (1, -1, 0) \times (0, 1, -1) = (1, 1, 1) = \overline{V}$

thus $\lambda = 1$

$F(u_1, u_2) \equiv 1$

thus G s.t. $F = \frac{\delta G}{\delta u_1} \Rightarrow G(u_1, u_2) = u_1$

thus $W = G\overline{\nabla} u_2 = u_1 \cdot (0, 1, -1)$

$= (x - y)(0, 1, -1)$

$\overline{W} = (0, x - y, y - x)$

(b) $\overline{V} = (x^2 + y^2,\ \overline{x^2 + y^2},\ 0)$

(1) $\overline{\nabla} \cdot \overline{V} = \frac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} + \frac{(x^2+y^2) - y \cdot 2y}{(x^2+y^2)^2} = 0$

(2) $\frac{dx}{\frac{x}{x^2+y^2}} = \frac{dy}{\frac{y}{x^2+y^2}} = \frac{dz}{0}$

$z = const,\ u_2 = z = c_2$

$lnx = lny + const,\ u_1 = \frac{x}{y}$

$\overline{\nabla} u_1 x \overline{\nabla} u_2 = (\frac{1}{y}, -\frac{x}{y^2}, 0) \times (0, 0, 1)$

$= (-\frac{x}{y^2}, -\frac{y}{y^2}, 0)$

$V = (\frac{-y^2}{x^2+y^2})(\overline{\nabla} u_1 x \overline{\nabla} u_2)$

$\Rightarrow \lambda(x, y, z) = \frac{-y^2}{x^2+y^2} = \frac{-1}{(\frac{x}{y})^2 + 1} = \frac{-1}{1+u_1^2}$

$F(u_1, u_2) \equiv \lambda(u_1, u_2) = \frac{-1}{1+u_1^2} = \frac{\delta G}{\delta u_1}$

thus, $G = -tan^{-1}(u_1)$

thus, $W = G\ \overline{\nabla} u_2 = -tan^{-1}(u_1) \cdot (0, 0, 1)$

$W = (0, 0, -tan^{-1}(\frac{x}{y}))$

Problem 4

To prove the existence and uniqueness of the equations, we need to show these equations satisfy conditions described in theorem 3.1.

Now, let us use parametrization that:

$x = t, y = y_0, z = f(t)$

Then $P(x, y, z)\frac{dy}{dt} - Q(x, y, z)\frac{dx}{dt}$

$= P(x, y, z)\frac{dy_0}{dt} = Q \cdot \frac{dt}{dt} = -1 \neq 0$ □

(2) Likewise parametrization:

$x = x_0, y = t, z = f(t)$

then $P(x, y, z) = 1$, in equation (3.9)

thus $P(x,y,z)\frac{dy}{dt} - Q(x,y,z)\frac{dx}{dt}$
$= 1 \cdot \frac{dt}{dt} - Q(x,y,z)\frac{dx_0}{dt} = 1 \neq 0$ □

Problem 5

Show that $u_2(x,y,z) = x^2 - (y^2+z^2)^{1/2} \cdot log\ \frac{(x^2+y^2)^{1/2}-z}{(x^2+y^2)^{1/2}+z}$ (Eq 2.15)
satisfies ($Eq2.14$).

Proof: $u_x = 2x$
$u_y = -\frac{1}{2}(y^2+z^2)^{-1/2} \cdot 2y \cdot f$
$- (y^2+z^2)^{1/2}\left\{\frac{\frac{1}{2}(y^2+z^2)^{-1/2}\cdot 2y}{(y^2+z^2)^{1/2}-z} - \frac{\frac{1}{2}(y^2+z^2)^{-1/2}\cdot 2y}{(y^2+z^2)^{1/2}+z}\right\}$
$= -(y^2+z^2)^{-1/2}yf - 2\frac{z}{y}$
$u_z = -(y^2+z^2)^{-1/2}2z \cdot f$
$- (y^2+z^2)^{1/2}\left\{\frac{\frac{1}{2}(y^2+z^2)^{-1/2}\cdot 2z-1}{(y^2+z^2)^{1/2}-z} - \frac{\frac{1}{2}(y^2+z^2)^{-1/2}\cdot 2z+1}{(y^2+z^2)^{1/2}+z}\right\}$
$= -z(y^2+z^2)^{-1/2} \cdot f - 2\frac{z^2}{y^2} + \frac{2(y^2+z^2)}{y^2}$
$= \frac{y^2+z^2}{y} \cdot 2x + xz \cdot (-(y^2+z^2)^{-1/2}yf - 2\frac{z}{y})$
$- xy(-z(y^2+z^2)^{-1/2} \cdot f - 2\frac{z^2}{y^2} + \frac{2(y^2+z^2)}{y^2})$
$= 0$
We have $u_1 = y^2 + z^2$
$\mathrm{u}_2 = x^2 - (y^2+z^2)^{1/2} log\ \frac{(x^2+y^2)^{1/2}-z}{(x^2+y^2)^{1/2}+z}$
where $V \cdot \overline{\nabla} u_1 = V \cdot \overline{\nabla} u_2 = 0$
$\overline{\nabla} u_1 = (0, 2y, 2z)$
$\overline{\nabla} u_2 = (2x, -(y^2+z^2)^{-1/2}f - 2\frac{z}{y}, -z(y^2+z^2)^{-1/2}f - \frac{2z^2}{y^2} + \frac{2(y^2+z^2)}{y^2})$
thus $\overline{\nabla} u_1 x \overline{\nabla} u_2 \neq 0$
(because if $\overline{\nabla} u_1 x \overline{\nabla} u_2 = 0$, then $u_1 // u_2$, but $0 \neq 2x$ for all $x \neq 0$
if $x = 0$, $f = log\frac{y-z}{y+z}$ and $u_2, y = -(y^2+z^2)^{-1/2} \cdot log\frac{y-z}{y+z} - \frac{z}{2y} \neq 2y(u_1, y)$ □)

$\Rightarrow u_1$ & $u_2 \Rightarrow$ fund independent □