Problem 1

Let $u(x, y, z)$ be a first integral of V and let C be an integral curve of V given by $x = x(t)$, $y = y(t)$, $z = z(t)$ $t \in I$.

Let us parameterize the first integral of V $u(x, y, z)$ by $u(x(t), y(t), z(t))$ on C and now take the derivative with respect to t.

$\frac{d}{dt}[u(x(t), y(t), z(t))] = \frac{du}{dx}\frac{dx}{dt} + \frac{du}{dy}\frac{dy}{dt} + \frac{du}{dz}\frac{dz}{dt}$

A level surface of $u(x(t), y(t), z(t))$ is such that $u(x(t), y(t), z(t)) = C_1$, or C and d$\overline{dtC_1=0}$

Then, we have

$\frac{du}{dx} + \frac{du}{dy}\frac{du}{dt} + \frac{du}{dz}\frac{dz}{dt} = 0$

But, this is equivalent to

$Pu_x + Qu_y + Ru_z = 0$ where $V =< P, Q, R >$

We see that $< P, Q, R > \cdot < u_x, u_y, u_z >= 0$

Therefore, the target vector of C (which is $V =< P, Q, R >$) was parallel to the level surface $u(x(t), y(t), z(t))$ for all $(x, y, z) \in \Omega$

Thus, we conclude that C must lie on some level surface of u for some C_1.

Problem 2

Find integral curves.

a) $V = (log(y + z), 1, -1)$

$\frac{dx}{log(y+z)} = \frac{dy}{1} = \frac{dz}{-1} \Rightarrow \int(1dy) = \int(-1dz)$

$y + z = c_1,$

$\frac{dy}{1} = \frac{dx}{log(y+z)} = \frac{dx}{log(c_1)} \Rightarrow \int log(c_1)dy = \int dx$

$log(c_1)y = x + c_2 \Rightarrow log(y + z)y - x = c_2$

$u_1(x, y, z) = y + z = c_1$ and $u_2(x, y, z) = log(y + z)y - x = c_2$

b) $V = (x^2, y^2, z(x + y))$, $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z(x+y)}$

$\int x^{-2}dx = \int y^{-2}dy \Rightarrow -x^{-1} = -y^{-1} + c_0$

$\frac{-y}{xy} = \frac{-x}{xy} + c_0 \Rightarrow \frac{x-y}{xy} = c_0$

$x = \frac{1}{\frac{1}{y}-c_0}$, $\frac{dy}{y^2} = \frac{dz}{z(x+y)} = \frac{dz}{z(\frac{1}{1/y-c_0}+y)} = \frac{dz}{z(\frac{y}{1-c_0y}+y)}$

$(\frac{1}{y-c_0y^2} + \frac{1}{y})dy = \frac{1}{z}dz;$

$(\frac{1-c_0y}{y-c_0y^2} + \frac{c_0}{1-c_0y} + 1/y)dy = (2/y + \frac{c_0}{1-c_0y})dy = \frac{1}{z}dz$

$\int(2/y + \frac{c_0}{1-c_0y})dy = 2 \cdot ln(y) - ln(1 - c_0y) = ln(\frac{y^2}{1-c_0y}) = ln(z) + c_1$

$\frac{y^2}{1-c_0y} = c_2z$, $\frac{y^2}{1-c_0y} = y \cdot \frac{y}{1-c_0y} = y\frac{1}{1/y-c_0} = xy$

$c_2z = xy$, $c_2 = xy/z$

$u_1(x, y, z) = \frac{x-y}{xy} = c_0$, $u_2(x, y, z) = \frac{xy}{z} = c_2$

Problem 3

Find the general solution

a) $\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$, then $\frac{dx}{y-z} + \frac{dy}{z-x} + \frac{dz}{x-y} = \frac{d(x+y+z)}{0}$

$x + y + z = c_1$

We also notice that if $P = y - z$, $Q = z - x$ and $R = x - y$, then $2xP + 2yQ + 2zR = 0$.

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There is a function X s.t.
$\frac{dX}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt}$
$X = x^2 + y^2 + z^2$, then $\frac{dx}{dt} = 2xP + 2yQ + 2zR = 0$
then $x^2 + y^2 + z^2 = c_2$
$u_1(x, y, z) = x + y + z = c_1$, $u_2(x, y, z) = x^2 + y^2 + z^2 = c_2$

Problem 4

a) $xu_x + yu_y + xy(z^2 + 1)u_z = 0$

Example 2.1 gives two first integrals that satisfy this equation.
$u_1(x, y, z) = y/x = c_1$, $u_2(x, y, z) = \frac{1}{2}xy - arctanz = c_2$
A general solution is a L' function $F(u_1, u_2)$ such that
$u = F(\frac{y}{x}, \frac{1}{2}xy - arctanz)$

b) $\frac{1}{y}(y^2 + z^2)u_x + xzu_y - xyu_z = 0$

Example 2.2 gives two first integrals that satisfy this equation
$u_1(x, y, z) = y^2 + z^2 = c_1$
$u_2(x, y, z) = x^2 - (y^2 + z^2)^{1/2} log\left[\frac{(y^2+z^2)^{1/2}-z}{(y^2+z^2)^{1/2}+z}\right] + c_2$
A general solution is a L' function $F(u_1, u_2)$ such that
$u = F(y^2 + z^2, x^2 - (y^2 + z^2)^{1/2} log\left[\frac{(y^2+z^2)^{1/2}-z}{(y^2+z^2)^{1/2}+z}\right])$

c) $(y + z)u_x + yu_y + (x - y)u_z = 0$

Example 2.3 gives two first integrals that satisfy this equation.
$u_1(x, y, z) = \frac{x+z}{y} = c_1$, $u_2(x, y, z) = (x - y)^2 - z^2 = c_2$
A general solution is a L' function $F(u_1, u_2)$ such that
$u = F(\frac{x+z}{y}, (x - y)^2 - z^2)$

d) $x(y - z)u_x + y(z - x)u_y + z(x - y)u_z = 0$

Example 2.4 gives the first integral $u_1(x, y, z) = xyz = c_1$
Simple inspection reveals that $u_2(x, y, z) = x + y + z = c_2$ is another functionally independent first integral.
A general solution is a L' function $F(u_1, u_2)$ s.t.
$u = F(xyz, x + y + z)$

Problem 5

Find the general solution

a) $u_{x1} = 0$ (n arbitrary)
A first integral is any function of the variables x_2, x_3, x_4, ..., x_n, so $u_1(x_1, x_2, ..., x_n) = x_2$, $u_2(x_1, x_2, ..., x_n) = x_3$, ..., $u_{n-1}(x_1, x_2, ..., x_n) = x_n$
Obviously these n-1 first integrals are all functionally independent.

A general solution is a L' function $F(u_1, u_2, u_3, ..., u_{n-1})$ s.t. $u = F(x_2, x_3, x_4, ..., x_n)$

b) $x_1 u_{x1} + x_1 x_2 u_{x2} + x_1 x_2 u_{x3} + x_4 u_{x4} = 0$ This implies $\frac{dx_1}{x_1} = \frac{dx_2}{x_1 x_2} = \frac{dx_3}{x_1 x_3} = \frac{dx_4}{x_4}$

so $\frac{dx_1}{x_1} = \frac{dx_4}{x_4} \Rightarrow lnx_1 = lnx_4 + c_0$

$\frac{x_1}{x_4} = c_1$, $grad = (\frac{1}{x_4}, 0, 0, -\frac{x_1}{(x_4)^2})$

$\frac{dx_2}{x_1 x_2} = \frac{dx_3}{x_1 x_3} \Rightarrow \frac{dx_2}{x_2} = \frac{dx_3}{x_3} \Rightarrow lnx_2 = lnx_3 + c$

$\frac{x_2}{x_3} = c_2$, $grad = (0, \frac{1}{x_3}, -\frac{x_2}{(x_3)^2}, 0)$

$\frac{dx_1}{x_1} = \frac{dx_2}{x_1 x_2} \Rightarrow dx_1 = \frac{dx_2}{x_2} \Rightarrow x_1 = lnx_2 + c$

$\frac{1}{x_2} e^{x_1} = c_3$, $grad = (\frac{1}{x_2} e^{x_1}, -e^{x_1} \frac{1}{(x_2)^2}, 0, 0)$

A general solution is a L' function $F(u_1, u_2, u_3)$ s.t. $u = F(\frac{x_1}{x_4}, \frac{x_2}{x_3}, \frac{1}{x_2} e^{x_1})$