Week 13

Outline

Taylor Series and Maclaurin Series Generated by a Function

–How to find the Taylor Series of a function in general

–Some Useful Taylor Series for $\frac{1}{1-x}$, e^x, $\sin x$

Convergence of Taylor Series

–Taylor's Theorem

–Remainder Estimation Theorem

More Examples

Applications of Taylor Series

–Approximating the Values of Functions

–Evaluating Nonelementary Integrals

–Evaluating Indeterminate Forms

Definition

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the Taylor series generated by f at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The Maclaurin series generated by f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots ,$$

the Taylor series generated by f at $x = 0$.

Remark

If two power series $\sum a_n(x-a)^n$ and $\sum b_n(x-a)^n$ are convergent and equal for all values of x in an open interval containing a, then $a_n = b_n$ for every n. Thus the Taylor series generated by f is unique. In particular, if f has a power series expansion at a, then it must be of the form

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

But there exist functions that are not equal to the sum of their Taylor series. For example,

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Direct computation shows that the Maclaurin series of f is 0.

Examples

Find the Taylor Series of the following functions.

1. $\frac{1}{1-x}$
2. e^x
3. $\sin x$
4. $\cos x$
5. $\ln(1+x)$
6. $\tan^{-1} x$

Definition

Let f be a function with derivatives of order k for $k = 1, 2, \cdots, N$ in some interval containing a as an interior point. Then for any integer n with $0 \leq n \leq N$, the Taylor polynomial of order n generated by f at $x = a$ is the polynomial

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Remarks

(1) The degree of $T_n(x)$ may be less that n since $f^{(n)}(a)$ may be zero.

(2) Of all polynomials of degree $\leq n$, the Taylor polynomial of order n gives the best approximation. More precisely, suppose that $f(x)$ is differentiable on an interval centered at $x = a$ and that $g(x) = b_0 + b_1(x-a) + \cdots + b_n(x-a)^n$ is a polynomial of degree at most n with constant coefficients $b_0, \ldots, b_n$. Let $E(x) = f(x) - g(x)$. If $E(a) = 0$ and $\lim_{x \to a} \frac{E(x)}{(x-a)^n} = 0$, then

$$g(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

In particular, $T_1(x)$ is the linearization of f at a.

Question

When is a function equal to the sum of its Taylor series?

Theorem

If $f(x) = T_n(x) + R_n(x)$, where $T_n(x)$ is the the Taylor polynomial of order n generated by f at a and

$$\lim_{x\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series in the interval $|x-a| < R$.

Example

Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, \text{ if } x \neq 0 \\ 0, \text{ if } x = 0. \end{cases}$$

Then the Taylor polynomial of order n generated by f at 0 is given by $T_n(x) = 0$. Thus $R_n(x) = f(x)$, which implies that $\lim_{x\to\infty} R_n(x) \neq 0$ if $x \neq 0$, that is, f is not equal to the sum of its Taylor series if $x \neq 0$.

Taylor's Theorem

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where

$$R_n(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, \mathrm{d}\, t,$$

or

$$R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some number c between a and x.

Remark

In particular, if $n = 0$, we have

$$f(x) = f(0) + R_0(x) = f(a) + f'(c)(x-a),$$

for some c between a and x. This is just the mean value theorem.

Question

How accurately do a function's Taylor polynomials approximate the function on a given interval?

The Remainder Estimation Theorem

If there is a positive constant M such that $|f^{(n+1)}(x)| \leq M$ for all $|x-a| \leq d$, then the remainder term $R_n(x)$ of the Taylor's series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \leq d$$

Examples

1. $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$
2. $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$
3. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$
4. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad |x| < \infty$
5. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots, \quad -1 < x \leq 1$
6. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad |x| \leq 1$
7. Find the Taylor series representations of the following functions.

(a) e^{x^2-2x+3} in power of $x-1$

(b) $\ln(3+x)$ about 3

(c) $x \ln x$ about 4

(d) xe^x about 2

(e) $\sin x$ about π

(f) $\cos^2 x$ about $\frac{\pi}{6}$

Approximating the Values of Functions

We can use Taylor polynomials (the partial sums of Taylor series) to approximate some functions. The problem is how accurate is the approximation. To estimate the error, we may apply Taylor's remainder estimation or Alternating series estimation theorem.

Examples

1. Estimate e with an error $< 10^{-6}$.
2. Estimate $e^{1.1}$ with an error $< 10^{-6}$.
3. For what values of x can we replace $\sin x$ by $x - \frac{x^3}{6}$ with an error of magnitude $< 3 \times 10^{-4}$.

4. Estimate $\cos 5^\circ$ with an error $< 5 \times 10^{-5}$.

5. Estimate $\tan^{-1}(0.2)$ with an error $< 5 \times 10^{-5}$.

6. In practice, to find the derivative of a function at a point, we can use the following two approximations.

$$f'(a) \approx \frac{1}{h}(f(a+h) - f(a)), \;\; f'(a) \approx \frac{1}{2h}(f(a+h) - f(a-h)),$$

where h is assumed to be very small. Both approximations are valid, but which one is better? To answer this question, do the following steps.

Step 1. Write down the first 4 terms of Taylor series for $f(x)$ about a.

Step 2. Rewrite the above Taylor series with $x = a + h$ and $x = a - h$.

Step 3. Plug in the result of Step 2 into the first approximation $\frac{1}{h}(f(a+h) - f(a))$ and simplify.

Step 4. Plug in the result of Step 2 into the second approximation $\frac{1}{2h}(f(a+h) - f(a-h))$ and simplify.

Step 5. Given the results in Steps 3 and 4, explain which approximation is better when $h \to 0$.

Evaluating Nonelementary Integrals

In general, it is hard to find the antiderivative of a function f. If f has the power series representation, then

$$\int f(x)\,\mathrm{d}\,x = \sum \int \frac{f^{(n)}(a)}{n!}(x-a)^n\,\mathrm{d}\,x,$$

provided the hypotheses of Term-by-Term Integration Theorem are satisfied.

Examples

6. Evaluate the following indefinite integrals as infinite series

(a) $\int e^{-x^2}\,\mathrm{d}\,x$

(b) $\int x\cos(x^3)\,\mathrm{d}\,x$

7. Use series to approximate the following definite integrals to within the indicated accuracy.

(a) $\int_0^1 e^{-x^2}\,\mathrm{d}\,x$ (three decimal places)

(b) $\int_0^1 x\cos(x^3)\,\mathrm{d}\,x$ (three decimal places)

Evaluating Indeterminate Forms

We can evaluate indeterminate forms by expressing the functions evolved as Taylor series.

For example, if $f(a) = g(a) = 0$, and f and g both have power series representations. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{\sum_{n=1}^{\infty} a_n (x-a)^n}{\sum_{n=1}^{\infty} b_n (x-a)^n}.$$

Usually, we only need the first several nonzero terms of the corresponding Taylor expansions.

Examples

8. $\lim_{x \to 0} \dfrac{x - \tan^{-1} x}{x^3}$

9. $\lim_{x \to 0} \dfrac{1 - \cos x}{1 + x - e^x}$

10. $\lim_{x \to 0} \dfrac{(e^x + e^{-x} - 2)\sqrt{5 + x^2 + \cos x}}{1 - \cos 5x}$

11. $\lim_{x \to 0} \dfrac{\cos(x^3) - 1 + \ln(1 + x^6)}{x^2(e^{x^2} - 1 - x^2)}$