Exercise 1.2.9. Find a recursive way to compute the number of spanning trees in a graph $\Gamma$.

Exercise 1.2.10. Let $M$ be a finite set. The partition complex $X$ is the flag complex whose vertices are non-trivial partitions of $M$ into two disjoint subsets (nontrivial means, none of the subsets is empty). There is an edge between $\{S_0, T_0\}$ and $\{S_1, T_1\}$ if these two partitions are nested, i.e, one of the four possible inclusions hold:

$$S_0 \subseteq S_1, \quad T_0 \subseteq S_1, \quad S_0 \subsetneq T_1, \quad T_0 \subsetneq T_1.$$ 

Prove that $X$ is a wedge of spheres.

1.3 Outer Space and its Relatives

1.3.1 Categories Based on Graphs

We already used a category of graphs above when we used Stallings folds. In that case, the morphisms were not allowed to crush edges. We shall have a need for at least two other categories based on the same class of objects (graphs) but with different sets of morphisms. But first, let us review a little bit about graphs.

We call a graph $\Gamma$ trivalent if every vertex has degree at least 3. A graph $\Gamma$ is called $n$-connected if simultaneously removing strictly fewer than $n$ edges will not disconnect $\Gamma$. (Removing an edge does not change the vertex set, we remove the interior only.)

Exercise 1.3.1. Let $\Gamma$ be $n$-connected, and let $v$ and $w$ be two vertices in $\Gamma$. Show that there are $n$ edge-disjoint paths from $v$ to $w$.

A 2-connected graph is called a core graph.
Exercise 1.3.2. Show that a graph is a core graph if and only if it is a union of reduced loops. (Recall that an edge path is reduced if it does not contain a subpath of the form \( \overrightarrow{e} \to \text{op}(\overrightarrow{e}) \).

Corollary 1.3.3. Every connected graph \( \Gamma \) has a core, i.e., a maximal core graph, which is unique; in fact, it is the union of all reduced closed edge paths in \( \Gamma \). \hspace{1cm} \text{q.e.d.}

Definition 1.3.4. Two edges \( e_0 \) and \( e_1 \) in a core graph \( \Gamma \) are equivalent if one of the following equivalent conditions is satisfied:

1. Either, removing both edges simultaneously disconnects \( \Gamma \); or \( e_0 = e_1 \).
2. Every reduced loop that passes through \( e_0 \) also passes through \( e_1 \).

From the first wording, it is apparent that equivalence is a reflexive and symmetric relation. From the second formulation, we infer that equivalence is transitive.

Exercise 1.3.5. Show that the conditions (1) and (2) are equivalent.

Based on the class of graphs, there is the category of collapses whose objects are graphs and whose morphisms are given by collapsing subforests: A collapse \( c: \Gamma_0 \to \Gamma_1 \) is an isotopy class of a map that sends edges either homeomorphically to edges or crushes them to points such that the preimage of the 0-skeleton of \( \Gamma_1 \) is a subforest of \( \Gamma_0 \).

Observation 1.3.6. Every collaps is a homotopy equivalence.

Observation 1.3.7. Vertex valency and graph connectivity can never decrease during a collaps. In particular any collaps of a trivalent core graph is a trivalent core graph.
1.3.2 Marked Graphs, Labelled Graphs, and Metric Trees

Exercise 1.3.8. Let $\Gamma_0$ and $\Gamma_1$ be two finite graphs with base points $P_0$ and $P_1$. Prove that every isomorphism

$$\phi : \pi_1(\Gamma_0, P_0) \to \pi_1(\Gamma_1, P_1)$$

is induced by a base point preserving homotopy equivalence $f : \Gamma_0 \to \Gamma_1$. Moreover, show that this homotopy equivalence $f$ is unique up to homotopy trough base point preserving maps.

Exercise 1.3.9. Show that a map $f : \Gamma_0 \to \Gamma_1$ between graphs is a homotopy equivalence if and only if it induces an isomorphism of fundamental groups.

Exercise 1.3.10. Let $f : \Gamma \to \Gamma$ be a self-homotopy equivalence of the base pointed finite graph $\Gamma$. Show that $f$ is homotopic (not relative to the base point!) to the identity if and only if $f$ induces an inner automorphism of $\pi_1(\Gamma)$.

Let $R$ be the rose with $n$ loops.

Definition 1.3.11. A **metric graph** is a finite graph $\Gamma$ together with an assignment of strictly positive real numbers (lengths) to its unoriented edges. The sum of of all these lengths is the **volume** of $\Gamma$.

A **marking** of $\Gamma$ is a homotopy equivalence

$$\mu : R \to \Gamma.$$

A **labelling** of $\Gamma$ is a homotopy equivalence

$$\lambda : \Gamma \to \Gamma.$$

Markings and labellings of base pointed graphs are supposed to preserve base points.
Two markings \( \mu_0 : R \to \Gamma_0 \) and \( \mu_1 : R \to \Gamma_1 \) of (metric) graphs are equivalent if there is an isomorphism (isometry) \( \zeta : \Gamma_0 \to \Gamma_1 \) such that the diagram

\[
\begin{array}{ccc}
\Gamma_0 & \xrightarrow{\zeta} & \Gamma_1 \\
\downarrow{\mu_0} & & \downarrow{\mu_1} \\
R & & R
\end{array}
\]

commutes up to homotopy (relative to base points if needed).

Similarly, two labellings \( \lambda_0 : \Gamma_0 \to R \) and \( \lambda_1 : \Gamma_1 \to R \) are equivalent, if there is an isomorphism (isometry) \( \zeta : \Gamma_0 \to \Gamma_1 \) such that the diagram

\[
\begin{array}{ccc}
\Gamma_0 & \xrightarrow{\zeta} & \Gamma_1 \\
\downarrow{\lambda_0} & & \downarrow{\lambda_1} \\
R & & R
\end{array}
\]

commutes up to homotopy (again base point preserving if there are base points involved).

**Observation 1.3.12.** There is a bijective correspondence of markings and labellings given by passage to a homotopy inverse. This correspondence is compatible with equivalence.

**Observation 1.3.13.** The group \( \text{Aut}(F_n) \), regarded as the mapping class group of \( R \) acts by composition on the set of marked graphs as well as on the set of labelled graphs. Both actions are compatible with equivalence. Markings and labellings, however, do not form isomorphic \( \text{Aut}(F_n) \)-sets: one of them is a right \( \text{Aut}(F_n) \)-set and the other one is a left-\( \text{Aut}(F_n) \) set, and switching from one side to the other involves inverting the group element.
1.3.3 Metric Trees and $\mathbb{R}$-Trees

**Definition 1.3.14.** A CAT($-\infty$)-space is called an $t$-tree $T$. This is a geodesic metric space wherein every geodesic triangle degenerates to a tripod. Let us expand this: For any two points $x, y \in T$, put

$$[x, y] := \{ z \in T \mid d(x, y) = d(x, z) + d(z, y) \}.$$ 

Then, $T$ is an $\mathbb{R}$-tree if the following conditions hold:

1. For all pairs $(x, y)$, the set $[x, y]$ is isometric to a segment in $\mathbb{R}$.

2. Whenever $[x, y] \cap [y, z] = \{ y \}$, we have $[x, z] = [x, y] \cup [y, z]$.

3. For any three point $x$, $y$, and $z$, there is a point $c$ such that

$$[x, y] \cap [x, z] = [x, c].$$

The **link** $\text{Lk}(x)$ of a $x$ is the set of infinitesimal geodesics issuing from $x$. More precisely, call two geodesic segments starting at $x$ equivalent if their intersection consists of more that $\{x\}$. Note that in this case, the intersection contains a whole non-trivial geodesic segment. A point in $\text{Lk}(x)$ is an equivalence class of geodesic segments starting at $x$. Note that we have a canonical map:

$$T - \{x\} \rightarrow \text{Lk}(x)$$

$$y \mapsto [x, y]$$

A point is called a **vertex** if its link does not contain precisely two elements. A vertex is called **terminal**, if its link is empty or contains one element.

**Exercise 1.3.15.** True or false: The elements of the link $\text{Lk}(x)$ correspond bijectively to the components of $T - \{x\}$. 

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Definition 1.3.16. Let $G$ be a group that acts on $T$ by isometries. The action is called minimal if $T$ does not contain a proper $G$-invariant subtree. The action is called reducible if one of the following holds:

1. Every element of $G$ is elliptic.

2. $T$ has an end that is stabilized by all of $G$.

3. The action of $G$ on the space of ends of $T$ has an invariant set consisting of precisely two ends. In this case, the action is hyperbolic.

Otherwise, the action is called irreducible. An action is semi-simple if it is trivial, hyperbolic, or irreducible.

The translation length

$$\tau : G \to \mathbb{R}^+$$

$$g \mapsto \min_{x \in T} d(x, gx)$$

of a given action is a length function because of the triangle inequality.

1.3.4 The Definition of Auter Space and Outer Space

Outer space, $\mathcal{X} := \mathcal{X}_n$, is a space acted upon by $\text{Out}(F_n)$. Its construction is based on metric graphs with a marking. Adding a base point to the graph, we will obtain a construction for auter space, $\mathcal{Y} := \mathcal{Y}_n$, upon which the group $\text{Aut}(F_n)$ acts. Let us describe the construction of $\mathcal{X}$.

The set $\mathcal{X} := \mathcal{X}_n$ of all equivalence classes of $R_n$-marked small volume 1 graphs (with base point) is outer space (auter space). We have to put a topology on this set. The idea is, of
course, that changing the length of edges slightly should not move you far in $\mathcal{X}$. However, we have to discuss the case where the length of an edge goes to 0 along a path: if this happens to a loop we run to infinity; if it happens to a non-loop, we move towards the collaps.

Before we embark on the topology of $\mathcal{X}$, let us give an alternative description of the underlying set. Let $\Gamma$ be a small metric graph of volume 1, and let $\mu : R \to \Gamma$ be a marking. The universal cover of $\Gamma$ is a \textbf{metric tree} $T$. The marking induces an action of $F_n = \pi_1(R)$ on $T$ by isometries.

**Definition 1.3.17.** A \textbf{length function} $\ell : G \to \mathbb{R}^+$ is a function satisfying

$$\ell(gh) \leq \ell(g) + \ell(h).$$


### 1.4 Proofs of Contractibility

**1.4.1 Proof by Continuous Folding (the Trees Proof)**

This is based on the work of M. Steiner and D. Skora.

**Definition 1.4.1.** A map $\varphi : \Gamma \to \Delta$ between metric graphs is a \textbf{piecewise isometry} if there is a decomposition of $\Gamma$ into line segments such that $\varphi$ is an isometry on each segment.

Let $\Gamma$ be a graph and $\Delta$ be a metric graph. Every graph morphism $f : \Gamma \to \Delta$ is homotopic relative to vertices to a piecewise isometry $\varphi$ for some suitably chosen metric on $\Gamma$ provided $f$. (Recall that graph morphisms do not crush edges.) The piecewise isometry $\varphi$ is called \textbf{tight} if it minimizes the length of each edge in $\Gamma$, i.e., every piecewise isometry $\psi$ that is homotopic to $\varphi$ relative to vertices does not allow for smaller edge lengths assigned to $\Gamma$.  

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(Note that we can consider each edge separately since we are dealing with homotopies that leave all vertices fixed.)

**Observation 1.4.2.** If a piecewise isometry of metric graphs is not locally injective, there is a continuous metric fold.

### 1.4.2 Proof by Sophisticated Low-Dimensional Topology (the Spheres Proof)

This is based on the work of A. Hatcher and K. Vogtmann.

Note that \( \pi_1(S^1 \times S^2) \) is infinite cyclic. Let \( M \) be a connected sum of \( n \) copies of \( S^1 \times S^2 \). Then \( F_n = \pi_1(M) \). Note that \( M \) consists of two \( n \)-handlebodies that are glued via the identity along their boundary surfaces. It is, therefore, easy to draw half of \( M \).

**Definition 1.4.3.** A **sphere set** is a finite set of disjoint, embedded spheres in \( M \). Such a set is **simple** if every complementary region is simply connected. Two sphere sets are **compatible** if their union is a sphere set.

A **sphere system** is a sphere set wherein no two spheres are isotopic and no sphere is trivial.

A **Dehn twist** in \( M \) along a sphere \( S \) with a specified axis is a diffeomorphism that is the identity outside a tubular neighborhood \( S \times [0,1] \) of \( S \) and that acts as a rotation by \( 2\pi t \) about the given axis in the slice \( S \times \{t\} \). Dehn twists change sphere systems only up to homotopy.

**Fact 1.4.4 (Laudenbach).** Homotopic sphere systems are isotopic.

**Corollary 1.4.5.** Dehn twists act trivially on \( Y \).
Fact 1.4.6 (Laudenbachs Dehn-Nielsen Theorem). The group Out($F_n$) is isomorphic to
\[ \pi_0(\text{Diff}^\circ(M)) \big/ \text{subgroup generated by Dehn twists} \]
and the isomorphism is given by the natural action of $\text{Diff}^\circ(M) = F_n$ on $\pi_1(M)$. In particular, there is a natural action of Out($F_n$) on $Y$.

Fix a simple sphere system $\mathcal{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$.

A. Hatcher figured out that two sphere systems can be isotoped as to minimize their intersections.

Definition 1.4.7. A sphere system $\mathcal{S}$ is in normal form with respect to $\mathcal{S}$, if for every sphere $S$ in $\mathcal{S}$, one of the following holds:

1. The sphere $S$ is contained in $\mathcal{S}$.

2. The sphere $S \cup \mathcal{S}$ is a sphere system.

3. The sphere $S$ has non-empty, transverse intersection with $\mathcal{S}$ and, for each component $W$ of $M - \mathcal{S}$ the following two conditions are both satisfied:

   (a) Each component of $S \cap W$ has at most one boundary circle in each boundary sphere of $W$.

   (b) No component of $S \cap W$ is isotopic in $W$ to a disk in the boundary of $W$.

Two sphere systems $\mathcal{S}_0$ and $\mathcal{S}_1$ in normal form are equivalent if there is a homotopy from $\mathcal{S}_0$ to $\mathcal{S}_1$ such that:

1. The common spheres of $\mathcal{S}_0$ and $\mathcal{S}$ stay fixed pointwise during the homotopy.

2. The homotopy is transverse to $\mathcal{S}$ on the other spheres at all times.
3. The circles in $S_1 \cap S$ vary only by an isotopy inside S. In particular, there is a well defined notion of innermost circles: A circle component $C$ in $S_0 \cap S$ is innermost if it bounds a disc $D$ in $S$ such that $D \cap S_0 = C$.

**Fact 1.4.8 (Hatcher).** Every sphere system can be isotoped as to be in normal form with respect to $S$; and any two isotopic sphere systems in normal form are equivalent.

### 1.4.3 A Continuous Contracting Flow

Let $\overline{X}$ be the simplicial complex whose $m$-simplices are isotopy classes of $(m+1)$-sphere systems. Note that these systems are not required to be simple. Let $Y$ be the simplicial complex whose vertices are isotopy classes of simple sphere systems and wherein a set of those systems forms a simplex if it is a chain (totally ordered) with respect to inclusion.

**Proposition 1.4.9.** The simplicial closure of outer space is isomorphic to $\overline{X}$ and the spine of outer space is isomorphic to $Y$.

**Proof.** !!! fix me !!! q.e.d.

**Construction 1.4.10 (Innermost Surgery).** Let $S$ be a sphere set that intersects $S$ transversally except for common spheres. Let $C$ be an innermost circle component in $S \cap S$, and let $S \in S$ be the sphere containing $C$. Consider a parallel copy of $S$ that intersects $S$ in an even smaller circle and perform surgery along the disc $D$, i.e., cut this copy along the disc and glue in discs to close the holes that are parallel to $D$. 
This way, we obtain to spheres $S_+$ and $S_-$. Note that $S \cup S_+ \cup S_- - S$ is a sphere set, compatible with $S$, that intersects $S$ in fewer circle components.

Obviously, we can surger simultaneously along several disjoint discs that are attached to the sphere $S$ from the same side.

**Observation 1.4.11.** Performing surgery in $S$ has the following effects on the complementary regions in $M - S$: The component that contains $D$ is cut along $D$ into two pieces. The component not containing $D$ is changed by attaching a 2-handle. Thus, if the complementary components were simply connected, they stay that way. In other words: If $S$ is simple, then innermost surgery yields simple sphere sets. \[ \text{q.e.d.} \]

A point in $\mathcal{X}$ is a formal convex combination of spheres that form a sphere system. The coefficient of a sphere can be thought of as a thickness assigned to the sphere. Spheres of width 0 are deleted from the picture. We will treat isotopy classes of sphere systems like sphere sets. Weights of parallel spheres are added up to give the weight of their isotopy class. We want to employ a continuous version of surgery.
Construction 1.4.12 (Continuous Parallel Innermost Surgery). Let $S = S_1 \cup \cdots \cup S_r$ with weights $t_1, \ldots, t_r$ satisfying $\sum t_i = 1$. Thicken those spheres $S_i$ a little bit that are not contained in the fixed system $\mathcal{S}$. We think of the thickened spheres as embedded annuli $S_i \times [0, t_i]$. Let $\overline{S}$ the sphere set obtained from $S$ by replacing $S_i$ by $S_i \times \{0\} \cup S_i \times \{t_i\}$. In $\overline{S}$, the weight $t_i$ is distributed evenly to both of these spheres. (Here, we use the convention that weights of parallel spheres add up.)

For each sphere $S$ in the fixed simple system $\mathcal{S}$, the intersection $\overline{S} \cap \mathcal{S}$ consists of disjoint circles, at most one for each sphere $S \in \mathcal{S}$. Let $T_{\overline{S}}$ be the metric tree whose vertices correspond to complementary regions in $\mathcal{S} - \overline{S}$ and whose edges correspond to those spheres in $\overline{S}$ that intersect $\mathcal{S}$. The length of an an edge in $T_{\overline{S}}$ is given be the weight of its corresponding sphere in $\overline{S}$.

We now perform innermost surgery simultaneously on all terminal points in the trees $T_{\overline{S}}$. We are doing continuous surgery and the weight is transferred to the surgered spheres to their successors so that the terminal edges in the trees $T_{\overline{S}}$ shrink with unit speed.

This defines a continuous flow on $\overline{\mathcal{X}}$.

Observation 1.4.13. The endpoint of each flow-line is a sphere system all of whose spheres are disjoint from the fixed system $\mathcal{S}$. Thus, if $\mathcal{S}$ is maximal, all spheres will be parallel to spheres in $\mathcal{S}$ at the end of the flow. In this case, the flow-lines all end in the simplex defined by $\mathcal{S}$, and the flow visibly defines a contraction. q.e.d.

Observation 1.4.14. It follows from (1.4.11) that the flow restricts to a flow on $\mathcal{X} \subset \overline{\mathcal{X}}$. It follows that Outer Space is contractible. q.e.d.
Lemma 1.4.18. Equivalence sphere systems $S_0$ and $S_1$ have identical combing paths.

Proof. The homotopy proving the equivalence is an isotopy in $S_1 \cap S$. Moreover, $S_1$ moves inside components of $M - S$. Therefore, the surgeries correspond bijectively. q.e.d.

Lemma 1.4.19. Let $S$ and $S'$ be two simple sphere systems, and suppose $S \subseteq S'$. Then the combing paths for $S$ and $S'$ are close.

Proof. It suffices to consider the case where $S$ is obtained from $S'$ by deleting one sphere. q.e.d.

Let $z$ be the complex whose $m$-simplices are $(m+1)$-systems of spheres. Let $z'$ be the union of all those simplices in $z$ that correspond to simple sphere systems. You should think of a point in $z$ as a sphere set where the spheres have a thickness and these weights add up to $1$.

Proposition 1.4.20. $z$ is contractible. The contraction induces a contraction of $z'$.

Proof. Now the edges in the trees have a thickness. You shrink them at unit speed from the terminal points. A sphere whose thickness becomes 0 is surged.

Finally, this contraction restricts to $z'$ because the 1-connectedness of the complement is preserved in the surgery process. q.e.d.

Consider a simplex $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_m$ in $\mathcal{S}$. Let us fix a sphere set $S$ and let $\overline{S}$ be a sphere set that contains two parallel copies for each sphere in $S$ each assigned an orientation pointing away from the other sphere. We assume $\overline{S}$ to be normal with respect to $S_m$. This implies that $\overline{S}$ is normal with respect to all $S_j$.