A morphism of graphs is a distance non-increasing map from the vertices of graph to the vertices of another graph. A folding of a graph is an idempotent graph endomorphism $f : \Gamma \rightarrow \Gamma$ such that the preimage of each vertex $v$ is either empty or contains precisely two vertices (one of which is $v$). The image $\alpha_f$ of a folding is called a half space or a root. Two foldings $f$ and $f'$ are opposite if their images are disjoint and the following hold:

$$f = f \circ f'$$
$$f' = f' \circ f.$$

**Exercise 3.1.** Show that a (locally finite) graph is the Cayley graph of a (finitely generated) Coxeter group if and only if the following conditions holds:

1. For each oriented edge $\overrightarrow{e}$ there is a unique folding $f_{\overrightarrow{e}}$ of $\Gamma$ satisfying $f_{\overrightarrow{e}}(\iota(\overrightarrow{e})) = \tau(\overrightarrow{e})$.
2. If $\overrightarrow{e}$ and $\overleftarrow{e}$ are opposite orientations of the same underlying geometric edge, then $f_{\overrightarrow{e}}$ and $f_{\overleftarrow{e}}$ are opposite foldings.

**Exercise 3.2.** Let $X$ be a complete metric space. Show that $X$ is geodesic if “it has midpoints”, i.e., for every pair $\{x, y\}$ there is a point $z$ such that $d(x, z) = d(y, z) = \frac{1}{2}d(x, y)$.

**Exercise 3.3.** Let $X$ be a complete metric space. Show that $X$ is a length space if “it has approximate midpoints”, i.e., for every pair $\{x, y\}$ and every $\varepsilon > 0$ there is a point $z$ such that $d(x, z), d(y, z) \leq \varepsilon + \frac{1}{2}d(x, y)$.

**Exercise 3.4.** Show that any two points in a CAT(0) space are connected by a unique geodesic segment.

Let $X$ be a CAT(0) space. A flat strip in $X$ is a convex subspace that is isometric to a strip in the Euclidean plane bounded by two parallel lines.

**Exercise 3.5 (Flat Strip Theorem).** Let $\gamma : \mathbb{R} \rightarrow X$ and $\gamma' : \mathbb{R} \rightarrow X$ be two geodesic lines. Show that the convex hull of these two lines is a flat strip provided that the geodesic lines are asymptotic, i.e., the function $d(\gamma(t), \gamma'(t))$ is bounded.

**Exercise 3.6.** Let $X$ be a connected complete metric space of curvature $\leq \kappa \leq 0$. Show that every free homotopy class has a representative that is a closed geodesic. Moreover, any two such representatives bound a “flat annulus”, i.e., they lift to bi-infinite geodesics in the universal cover that bound a flat strip.

Each problem is worth 5 points, but you can earn at most 20 points with this assignment.

Late homework will not be accepted.