Exercise 1.1. Prove that a gallery from $C$ to $D$ has minimum length if and only if it does not cross any hyperplane twice. Moreover, the set of hyperplanes that are crossed by a minimum length gallery from $C$ to $D$ is precisely the set of those $H \in \mathcal{H}$ that separate $C$ from $D$. In particular, this set is the same for all those minimum length galleries.

Exercise 1.2. Show that for any $s, t \in S$,

$$\langle u_s, u_t \rangle = \begin{cases} -\cos \left( \frac{\pi}{m_{s,t}} \right) & \text{for } m_{s,t} \text{ finite} \\ -1 & \text{for } m_{s,t} \text{ infinite} \end{cases}$$

Exercise 1.3. Show that the following are equivalent:

1. $\mathcal{H}$ is finite.
2. $W$ is finite.
3. $W$ is torsion.
4. $\bigcap_{H \in \mathcal{H}} H \neq \emptyset$.

Definition. Let $G = \langle x_1, \ldots, x_r \rangle$ be a group generated by a given finite generating set $\Sigma = \{x_1, \ldots, x_r\}$.

The word problem for the pair $(G, \Sigma)$ is the following:

Is there an algorithm that takes as input any word $w$ over $\Sigma \cup \Sigma^{-1}$ and prints “yes” if the word $w$ evaluates in $G$ to 1 and prints “no” otherwise?

If there is such an algorithm, the pair $(G, \Sigma)$ is said to have solvable word problem.

The conjugacy problem for the pair $(G, \Sigma)$ is the following:

Is there an algorithm that takes as input any pair $(w, u)$ of words over $\Sigma \cup \Sigma^{-1}$ and prints “yes” if the two word evaluate to conjugate elements of $G$ and prints “no” otherwise.

If there is such an algorithm, the pair $(G, \Sigma)$ is said to have solvable conjugacy problem.

Exercise 1.4. Let $\Sigma$ and $\Xi$ be two finite generating sets for $G$. Show that the pair $(G, \Sigma)$ has a solvable word problem (conjugacy problem) if and only if the pair $(G, \Sigma)$ has a solvable word problem (conjugacy problem). As the parentheses indicate, there is a proof that works for both problems.
Exercise 1.5. Let $H \hookrightarrow G \rightarrow F$ be a short exact sequence of groups where $F$ is finite and $H$ is finitely generated and has solvable word problem. Show that $G$ is finitely generated and has solvable word problem.

Exercise 1.6. Let $H$ be a subgroup of finite index in the finitely generated $G$. Then $H$ is finitely generated, too. Show that $H$ has solvable word problem if and only if $G$ has solvable word problem.

Each problem is worth 5 points, but you can earn at most 20 points with this assignment.

Late homework will not be accepted.