B Topology: Nuts and Bolts

B.1 Topological Categories

There are way too many topological spaces – most of them pathological. Hence real topology (as opposed to point set topology) takes place in smaller categories. The additional structure makes their objects amenable to stronger methods. Fortunately, all spaces we are interested in, are very nice.

B.1.1 Paracompact Spaces

A Hausdorff space X is paracompact if every open cover has a locally finite subcover. Any space you would want to meet is paracompact. It is the minimum requirement for a space to be considered "nice" in any sense. The technical importance of paracompactness is

Fact B.1. *If X is paracompact and $\mathcal{U}$ is an open cover, then there is a partition of unity subject to $\mathcal{U}$.*

Example B.2. All manifolds are paracompact as they have a countable basis. All CW-complexes are paracompact.

B.1.2 Complexes

Cell Complexes come in different flavors. The most general kind is build from a set of vertices by successively glueing in higher dimensional cells. The cells of dimension $m+1$ are balls whose boundary spheres are mapped via attaching maps into the m-skeleton which has been constructed already. The generality lies in the fact, that we do not make any assumptions about the attaching maps.

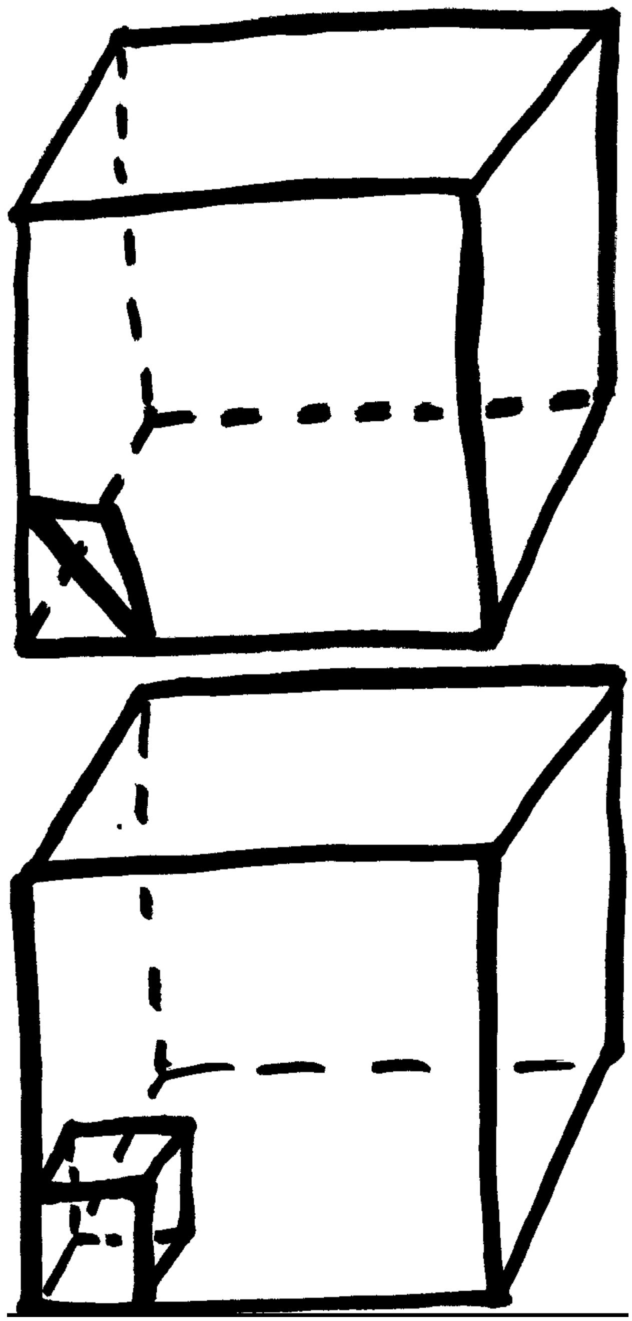
Piecewise Euclidean cell complexes have cells that have the additional structure of convex polyhedra in Euclidean space. Here the attaching maps are supposed to identify a boundary cell of an $(m+1)$-cell isometrically with a cell in the m-skeleton, which then is called a face of the $(m+1)$-cell. The category of piecewise

Euclidean cell complexes is suitable for geometric methods because broken straight line paths have lengths: you can measure the lengths of the pieces inside the cells.

Generalized simplicial complexes are piecewise Euclidean cell complexes all of whose cells are regular unit length simplices. They differ from ordinary simplicial complexes only in that two simplices might share more than one boundary simplex. For instance you can realize a sphere simply by gluing two simplices along their boundary. The result is a generalized simplicial complex which is not a simplicial complex. Thus, we call a piecewise Euclidean cell complex combinatorial if the intersection of any two closed cells is empty or consists of just one closed cell. Note that links in combinatorial complexes are combinatorial.

A simplicial complex is called a flag complex if it does not have "hollow simplices", i.e., if you see the one skeleton of a triangle, there is a 2-simplex filled in; if you see the 2-skeleton of a tetrahedron, there is a 3-simplex filled in; and so on. Observe that any barycentric subdivision of a combinatorial cell complex is a flag complex.

Another useful specialization of piecewise Euclidean complexes are cube complexes whose cells look like unit cubes. Combinatorial cube complexes are more combinatorial than topological objects — very much like simplicial complexes. Compared to simplicial complexes they have the advantage that the cross product of cube complexes is a cube complex. Moreover, you can define links in cube complexes in two ways: in one way the links are simplicial complexes in the other way the links are cube complexes themselves.



It turns out that for geometry, the links a better regarded as simplicial complexes – in fact, the simplices should be viewed metrically as spherical simplices.

B.1.3 Posets

A combinatorial substitute for cell complexes is the notion of a poset (partially ordered set). Every cell complex gives rise to a poset: The elements of the posets are the cells of the complex and the order is given by the face relation. Employing this analogy, let us agree to call the elements of a poset (open) cells. Moreover, if we have two cells p and q satisfying $p \preceq q$, we say that p is a face of q or, equivalently, that q is a coface of p. If $p \preceq q$ with $p \neq q$, we say that p is a strict face of q. The boundary of a cell is the subposet of all its strict faces, and the link is the subposet of all its strict cofaces. Thus, the link should properly be called the coboundary.

For each poset, the finite totally ordered subsets (chains) form a simplicial complex, whose geometric realization is considered the geometric realization of the poset. This way, all topological notions apply to posets. The analog of a closed cell in a poset is the subposet formed by a cell together with its boundary. The analog of a closed subset is a closed subposet, that is, a subposet that contains all faces of any cell it contains. I never encountered someone using the dual notion of a coclosed subposet.

Let us denote by P^{op} the poset P with the order relation reversed. Observe that P and P^{op} have the same geometric realization as a finite totally ordered subset of one is also a totally ordered subset of the other.

A morphism $f : P \to Q$ of posets is just a map that preserves the partial ordering. Given a cell $q \in Q$, we have define the fibre of f over q to be the subposet

$$f/q := \{p \in P \mid f(p) \preceq q\}$$

and the cofibre to be

$$q\backslash f := \{p \in P \mid q \preceq f(p)\}.$$

B.2 Computing Homotopy Groups

A space X is called <u>m-connected</u> if, for every $i \leq m$ any map from $\mathbb{S}^i \to X$ extends to a map $\mathbb{B}^{i+1} \to X$. Note that -1-connected means non-empty and that 0-connected is the same as path-connected and non-empty.

B.2.1 Combinatorial Morse Theory

Let X be a piecewise Euclidean cell complex. A <u>Morse function</u> on X is a map $h : X \to \mathbb{R}$ that is affine on closed cells and satisfies the following slope condition:

> There is an $\varepsilon > 0$ such that
>
> $$\varepsilon < |h(w) - h(v)|$$
>
> for all pairs of vertices v and w joined by an edge in X.

Note that h is, in particular, non-constant on edges.

We think of $h(x)$ as the height of the point $x \in X$. For each vertex v, we define the <u>descending link</u> $\mathrm{Lk}^{\downarrow}(v)$ with respect to h to be that part of $\mathrm{Lk}(v)$ spanned by all the cells that contain v as a vertex of maximum height. For each height s, define the sublevel set $X_s := \{x \in X \mid h(x) \leq s\}$.

Morse Lemma. *For any two heights $s < t$ with $t - s < \varepsilon$*

$$X_t \backsimeq X_s \cup_D C$$

where

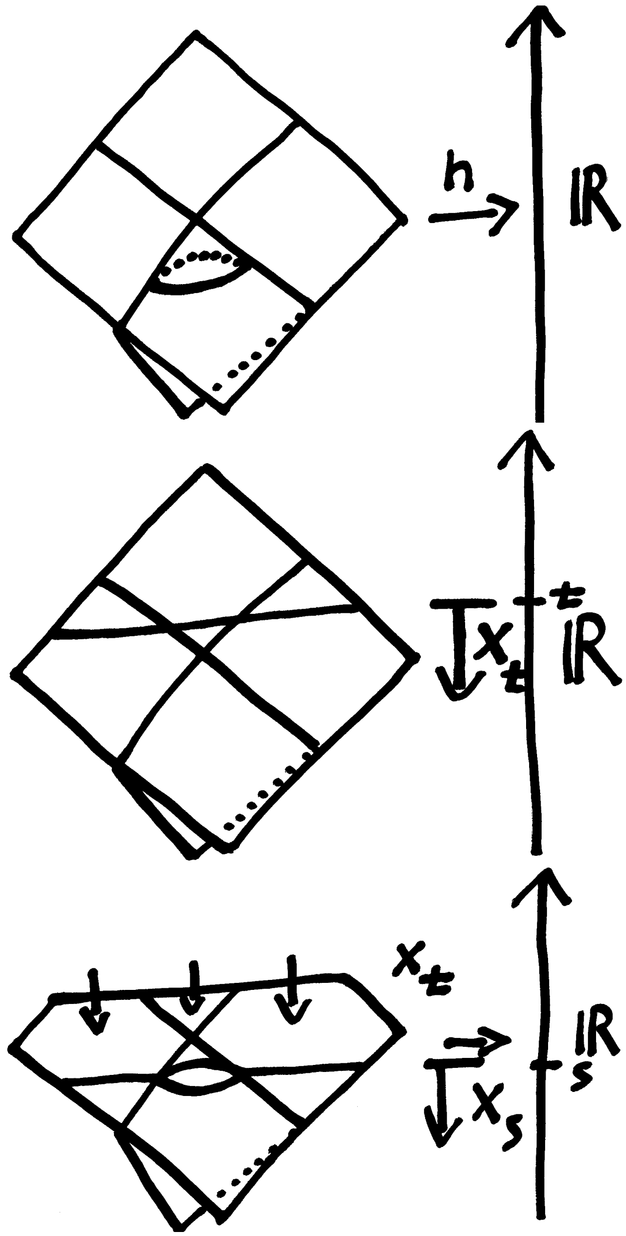
$$D := \biguplus_{s<h(v)\leq t} \mathrm{Lk}^{\downarrow}(v)$$

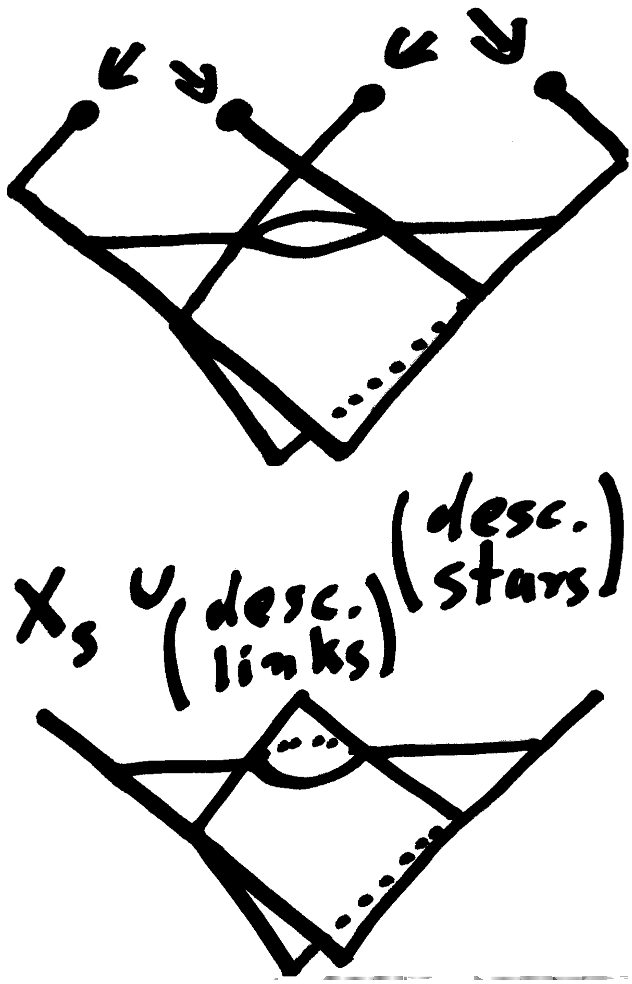
and

$$D := \biguplus_{s<h(v)\leq t} \mathrm{Cone}\left(\mathrm{Lk}^{\downarrow}(v)\right).$$

In words: In order to get X_t from X_s up to homotopy equivalence, you have to cone off the descending links of the vertices that are in $X_t - X_s$.

Proof. Here is a sequence of pictures:





All we do is pushing in free faces. This defines a deformation retraction. **q.e.d.**

Corollary B.3. *If all descending links are contractible, then the homotopy type of sublevel sets does not change as the height increases. Hence the whole space has the homotopy type of any of its sublevel sets.* **q.e.d.**

B.2.2 CAT(0) (Cube) Complexes

A geodesic metric space X is called CAT(0) if every geodesic triangle Δ in X compares to a Euclidean triangle Δ', i.e.: Δ' is a triangle in the Euclidean plane that has the same side length as Δ whence we have a notion of a "corresponding point". Then for any two point in Δ, their distance is bounded from above by the distance of their corresponding points.

In general, a geodesic metric space is called CAT(κ) if every geodesic triangle compares to a triangle in the contractible two dimensional Riemannian manifold of constant curvature κ. Of particular importance are CAT(1), CAT(-1), and the aforementioned CAT(0).

Example B.4. Any Riemannian manifold with non-positive sectional curvature is CAT(0).

Fact B.5 ([BrHa99, Theorem I.7.13]). *If the cells of a piecewise Euclidean complex X form only finitely many shapes, then X is a complete geodesic metric space.*

The Cartan-Hadamard theorem states that Riemannian manifolds with non-positive sectional curvature have unique geodesics if they are simply connected. This generalizes, in fact, to complete CAT(0) spaces and implies

Fact B.6. *Any complete CAT(0)-space is contractible. Any locally CAT(0)-space has a universal cover that is CAT(0).*

Of particular importance are combinatorial cube complexes, because here we have a criterion for being locally CAT(0).

Gromov's Lemma. *A combinatorial cube complex is locally CAT(0) if and only if every vertex link is a flag complex.*

B.2.3 The Vietoris-Smale-Quillen Argument

Lifting a theorem of Vietoris from homology to homotopy, S. Smale proved:

Fact B.7 ([Smal57]). *Let X and Y be 0-connected, locally compact, Hausdorff and metrizable. Let $f : X \to Y$ be proper and onto. Assume that X is locally m-connected, that Y is locally $(m-1)$-connected,* *and that $f^{-1}(y)$ is locally $(m-1)$-connected, for any point $y \in Y$. Then the following hold:*

1. *Y is locally m-connected.*

2. *The map f induces an isomorphism in homotopy groups*

$$\pi_i(X) \to \pi_i(Y)$$

   for $0 \leq i < m$ and an epimorphism for $i = m$.

For geometric group theory the combinatorial setting is more suitable. The following version, due to D. Quillen, is stated in the language of posets. Note that it therefore applies to all piecewise Euclidean cell complexes.

Fact B.8 ([Quil78]). *Let $f : P \to Q$ be a morphism of posets. Assume that each fibre is m-connected. Then P is m-connected if and only if Y is m-connected.*

Moreover, if each fibre is contractible, then f is a homotopy equivalence.

This is a very powerful tool for computing the connectivity of a space.

B.2.4 Nerves

Definition B.9. The nerve of a cover $(U_i)_{i\in I}$ is the simplicial complex

$$N := \left\{ \sigma \subset I \,\middle|\, \bigcap_{i\in\sigma} U_i \neq \emptyset \right\}.$$

Fact B.10. *Let X be a paracompact topological space and $\mathcal{U} = (U_i)_{i\in I}$ be a cover by open subsets. If for every simplex σ in the nerve N, the subset*

$$X_\sigma := \bigcap_{i\in\sigma} U_i$$

is a contractible subspace of X, then X and N are homotopy equivalent.

Fact B.11. *Let X be a locally compact CW-complex and $\mathcal{U} = (U_i)_{i \in I}$ be a cover by closed subcomplexes. If for every simplex σ in the nerve N, the subcomplex*

$$X_\sigma := \bigcap_{i \in \sigma} U_i$$

is contractible, then X and N are homotopy equivalent.

Fact B.12. *Let P be a poset $\mathcal{U} = (U_i)_{i \in I}$ be a cover by closed subposets. If for every simplex σ in the nerve N, the subposet*

$$P_\sigma := \bigcap_{i \in \sigma} U_i$$

is contractible, then P and N are homotopy equivalent.

We give a proof of the poset-version to illustrate the connection with the Vietoris-Smale-Quillen argument.

Proof. Consider the map

$$\begin{array}{rcl} P & \xrightarrow{f} & N^{\mathrm{op}} \\ p & \mapsto & \{ i \in I \mid p \in U_i \} . \end{array}$$

The fibres are easily seen to coincide with the subposets U_σ. Thus, the fibres are contractible, and f is a homotopy equivalence by (B.8). **q.e.d.**