

# A How to Specify a Group

## A.1 Generators and Relations

### A.1.1 Generating Sets / Cayley Graphs

**Definition A.1.** Let  $S$  be a set of elements in the group  $G$ . The intersection of all subgroups  $H \leq G$  containing  $S$  is a subgroup of  $\langle S \rangle \leq G$ . It is called the subgroup generated by  $S$ . The subset  $S$  is called a generating set for  $\langle S \rangle$ .

**Definition A.2.** A graph  $\Gamma$  is a map  $\tau: \vec{\mathcal{E}}_\Gamma \rightarrow \mathcal{V}_\Gamma$  where  $\mathcal{V}_\Gamma$  is a set (its elements are called vertices) and  $\vec{\mathcal{E}}_\Gamma$  is a free  $\mathbb{Z}_2$ -set, i.e., a set together with a fixpoint free involution  $\text{op}: \vec{\mathcal{E}}_\Gamma \rightarrow \vec{\mathcal{E}}_\Gamma$ . The elements of  $\vec{\mathcal{E}}$  are (oriented) edges. The elements of  $\mathcal{E}_\Gamma := \vec{\mathcal{E}}_\Gamma / \text{op}$  are called (geometric) edges.

An orientation on  $\Gamma$  is a section  $\circ: \mathcal{E} \rightarrow \vec{\mathcal{E}}$ .

We define another map  $\iota: \vec{\mathcal{E}}_\Gamma \rightarrow \mathcal{V}_\Gamma$  by  $\iota(\vec{e}) := \tau(\vec{e}^{\text{op}})$ . The map  $\tau$  assigns to each edge its terminal vertex whereas  $\iota$  provides the initial vertex of the edge.

**Definition A.3.** A  $G$ -labeling of a graph  $\Gamma$  is a map  $\phi: \vec{\mathcal{E}}_\Gamma \rightarrow G$  satisfying

$$\phi(\vec{e}^{\text{op}}) = \phi(\vec{e})^{-1}.$$

**Definition A.4.** Let  $G$  be a group with finite generating system  $\Sigma$ . The (left) Cayley graph  $\Gamma := \Gamma_\Sigma(G)$  is constructed as follows: The set  $\mathcal{V}_\Gamma$  is  $G$ . The set of oriented edges is  $\vec{\mathcal{E}}_\Gamma := G \times \Sigma \times \{\pm 1\}$ . We have to specify a fixpoint free involution  $\text{op}: \vec{\mathcal{E}}_\Gamma \rightarrow \vec{\mathcal{E}}_\Gamma$  and an endpoint map  $\tau$ . For any oriented edge  $\vec{e} = (g, \sigma, \varepsilon)$

$$\begin{aligned}\vec{e}^{\text{op}} &:= (g\sigma, \sigma^{-1}, -\varepsilon) \\ \tau(\vec{e}) &:= g\end{aligned}$$

The set of geometric edges is obviously isomorphic to  $G \times \Sigma$  and the map

$$(g, \sigma) \mapsto (g, \sigma, 1)$$

defines an orientation on  $\Gamma$ . We regard this as the standard or positive orientation. Note that  $G$  acts from the left on  $\Gamma_\Sigma(G)$ , and this action preserves the orientation.

There is a corresponding notion of a right Cayley graph upon which  $G$  acts from the right.

For an oriented edge  $\vec{e} = (g, \sigma, \varepsilon)$ , we call  $\sigma$  is the label of  $\vec{e}$ . We denote the label of  $\vec{e}$  by  $\sigma_{\vec{e}}$ . When we talk about labels of geometric edges, we either (silently) identify them with the positively oriented edges or there is an implied orientation for the edge (e.g., when the edge is part of a directed path).

Any directed path in  $\Gamma$  reads a word in  $\Sigma \uplus \Sigma^{-1}$ : while you are moving along the path, you pick up the labels of the oriented edges you are traveling. Note that all  $G$ -translates of a given path read the same word. On the other hand, any word  $w$  over  $\Sigma \uplus \Sigma^{-1}$  defines a  $G$ -orbit of paths: For any vertex in  $v \in \Gamma$  there is a unique directed path starting at  $v$  that reads  $w$ .

### A.1.2 Defining Relations / Cayley Complexes

**Definition A.5.** Let  $G = \langle \Sigma \rangle$  be a group and  $\Gamma = \Gamma_\Sigma(G)$ . A Relation in  $G$  over  $\Sigma$  is a word  $w$  over  $\Sigma \uplus \Sigma^{-1}$  that evaluates to  $1 \in G$  by multiplication of its letters.

Note that relations in  $G$  correspond to loops in  $\Gamma$ . Therefore, a relation  $r$  over  $\Sigma$  determines a  $G$ -orbit of loops in  $\Gamma$ . We can  $G$ -equivariantly glue in a family  $\mathcal{D}_r$  of 2-cells killing theses loops: The 2-cells are polygons whose boundaries are reading  $r$ .

A set  $R$  of relations defines  $G$  if glueing in  $\bigcup_{r \in R} \mathcal{D}_r$  kills the fundamental group of  $\Gamma$ . In this case, the resulting space

$$\Gamma_{\Sigma, R}(G) := \Gamma \cup \bigcup_{r \in R} \mathcal{D}_r$$

is called the Cayley complex of the presentation  $\mathcal{P} = \langle \Sigma \mid R \rangle$  which is said to present the group  $G$ .

Note that  $G$  acts freely (from the left) on  $\Gamma_{\Sigma, R}$  and the

quotient

$$K_{\mathcal{P}} := G \backslash \Gamma_{\Sigma, R}(G)$$

is called the presentation complex or standard two-complex of the presentation  $\mathcal{P}$ . Moreover,  $\Gamma_{\Sigma, R}(G)$  is the universal cover of  $K_{\mathcal{P}}$  and  $G$  acts as the group of deck transformations. In particular, every group is the fundamental group of a 2-complex since

$$G = \pi_1(K_{\mathcal{P}}).$$

Two presentations are equivalent if they present the same group (or more precisely, if they present isomorphic groups).

**Fact A.6.** *Let  $\Sigma$  generate  $G$ . A set  $R$  of words over  $\Sigma \uplus \Sigma^{-1}$  defines  $G$  if and only if it generates the kernel of the canonical projection*

$$F_{\Sigma \uplus \Sigma^{-1}} \rightarrow G$$

*as a normal subgroup of  $F_{\Sigma \uplus \Sigma^{-1}}$ .*

**Fact A.7.** *Let  $\langle \Sigma \mid R \rangle$  be a presentation for  $G$ . Then, a word  $w$  is a relation in  $G$  over  $\Sigma$  if and only if, when interpreted as an element of the free group  $F_{\Sigma \uplus \Sigma^{-1}}$ , it represents an element of the normal subgroup generated by  $R$ .*

**Fact A.8.** *The “calculus of presentation” is given by the following rules.*

1. *Let  $\langle \Sigma \mid R \rangle$  present  $G$ , and let  $R'$  be a set of relations in  $G$  over  $\Sigma$ . Then*

$$\langle \Sigma \mid R \cup R' \rangle$$

*also presents  $G$ . This process is called adding redundant relations. There is an obvious inverse process of deleting redundant relations.*

2. Let  $\langle \Sigma \mid R \rangle$  present  $G$ , and let  $w$  be a word over  $\Sigma \uplus \Sigma^{-1}$  that represents the element  $g \in G$ , then

$$\langle \Sigma \cup \{g\} \mid R \cup \{wg^{-1}\} \rangle$$

also presents  $G$ . This process is called adding a generator and a defining equation. You can also add many generators with their defining equations at the same time – even infinitely many.

The inverse passage from a presentation of the form

$$\langle \Sigma \cup \{g\} \mid R \cup \{wg^{-1}\} \rangle$$

to  $\langle \Sigma \mid R \rangle$  is the deletion of a redundant generator. Again, you can delete many generators at the same time.

3. Any two presentations are equivalent if and only if you can pass from one to the other by a finite chain of the following processes: adding redundant relations, deleting redundant relations, adding generators and defining equations, deleting redundant generators.

**Remark A.9.** A relation of the form  $wg^{-1}$  is often written (and always read) as  $w = g$  for this is what it says.

**Exercise A.10.** Show that the following presentations define isomorphic groups:

$$F_{\text{finite},a} := \langle a, b \mid b^{aa} = b^{ab}, b^{abb} = b^{abb^a} \rangle$$

$$F_{\text{finite},b} := \langle c, d \mid d^{cc} = d^{cd}, d^{cdc} = d^{cdd} \rangle$$

$$F_{\text{infinite}} := \langle x_0, x_1, x_2, \dots \mid x_i^{x_j} = x_{i+1} \text{ for } j < i \rangle.$$

Here, we use the convention  $g^h := h^{-1}gh$ .

Let us discuss the question of how to come up with presentations. In addition to ad hoc methods (which are at times unavoidable and can be very powerful too), there are two geometric principals.

**Theorem A.11.** *Let  $G$  act by homeomorphisms on the 1 connected space  $X$ , and let  $U$  be an open 0-connected subset in  $X$  such that  $X = GU$ . Put*

$$\Sigma := \{g \in G \mid U \cap gU \neq \emptyset\}$$

and

$$R := \{xy = (xy) \mid x, y \in \Sigma \text{ and } U \cap xU \cap xyU \neq \emptyset\}.$$

Then  $\mathcal{P} := \langle \Sigma \mid R \rangle$  is a presentation for  $G$ .

**Proof.** Let  $\tilde{G}$  be the group presented by  $\mathcal{P}$ . Define  $\tilde{X} = \bigsqcup_{g \in \tilde{G}} gU / \sim$  where

$$g_0u_0 \sim g_1u_1 : \iff g_0 = g_1x, u_1 = xu_0 \in U \cap xU \text{ for some } x \in \Sigma.$$

Let us call  $x$  a witness for the equivalence.

It is not hard to show that  $\sim$  is an open equivalence relation. For transitivity, you have to use the defining relations for  $\tilde{G}$ : If

$$g_0u_0 \sim g_1u_1$$

with witness  $x \in \Sigma$  and

$$g_1u_1 \sim g_2u_2$$

with witness  $y \in \Sigma$ , then

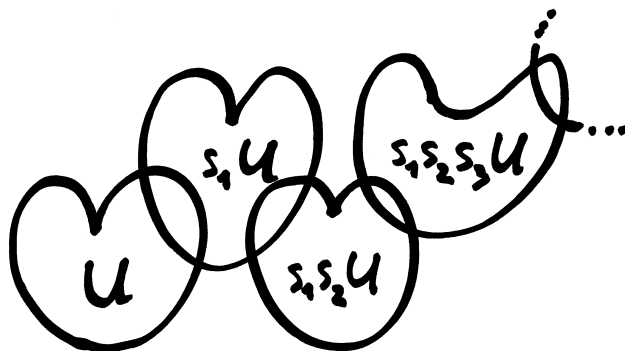
$$u_0 = xu_1 = xyu_2 \in U \cap xU \cap xyU \neq \emptyset$$

whence  $(xy) \in \Sigma$  and  $R$  contains the relation  $xy = (xy)$ . Then

$$g_0u_0 \sim g_2u_2$$

with witness  $(xy)$ .

Since  $U$  is path-connected, so is  $\tilde{X}$ . This is clear from the picture:



There is an obvious group homomorphism  $\tilde{G} \rightarrow G$  which induces a map  $\tilde{X} \rightarrow X$ . This map is a covering map, and the group of deck-transformations is  $\ker(\tilde{G} \rightarrow G)$ . Since  $X$  is 1-connected and  $\tilde{X}$  is a connected cover, the projection is an isomorphism, the group of deck-transformations is trivial, and  $\tilde{G} = G$ . **q.e.d.**

The second geometric source of presentations are 2-complexes. Their fundamental groups have presentations that can be read off the complex easily. This method goes way back to Poincare, was described in a more precise fashion by Tietze, and works because of the Seifert-Van-Kampen theorem.

**Fact A.12.** *Given a 2-complex  $X$ , a presentation for  $\pi_1(X)$  can be read off as follows:*

1. *Choose a spanning tree  $T$  for the 1-skeleton.*
2. *Introduce a formal generator for each edge in  $X$ .*
3. *For each 2-cell read off a relation along its boundary.*
4. *Finally declare trivial each generator that comes from an edge in  $T$ .*

Let  $H \leq G = \langle \Sigma \mid R \rangle$ . Then  $H$  is the fundamental group of a cover of the presentation 2-complex associated to the presentation  $\mathcal{P} = \langle \Sigma \mid R \rangle$ . In this case (A.12) is known as the Reidemeister-Schreier method for finding presentations of subgroups of groups defined by generators and relations.

**Fact A.13.** Let  $H \leq G = \langle \Sigma \mid R \rangle$ , and fix a set  $T$  of words  $w_1, \dots, w_r$  in the free group  $F_\Sigma$  representing the right cosets of  $H$  in  $G$ . (Such a set is called a Schreier transversal.) For each  $g \in G$ , let  $\bar{g}$  be the word  $w_i$  representing the coset  $Hg$ .

For each  $w \in T$  and  $x \in \Sigma$ , define

$$y_{w,x} := wx(\overline{wx})^{-1}$$

Then  $H$  is generated by the  $y_{w,x}$  with defining relations  $wrw^{-1}$  where  $w \in T$  and  $r \in R$ .

This needs an illustration.

**Example A.14.** Let  $H$  be the kernel of the homomorphism

$$\langle x_1, \dots, x_r \mid x_1^2 \cdots x_r^2 = 1 \rangle \rightarrow C_2$$

sending each  $x_i$  to the non-trivial element.

There are two cosets, represented by 1 and  $x_r$ . We have the following generators for  $H$ :

$$\begin{aligned} y_i &:= 1x_i x_r^{-1} && \text{for } i \leq r-1 \\ z_i &:= x_r x_i && \text{for } i \leq r. \end{aligned}$$

Now, we write the two relations first in the generators  $x_i$ :

$$\begin{aligned} x_1^2 \cdots x_r^2 &= 1 \\ x_r x_1^2 \cdots x_r^2 x_r^{-1} &= 1. \end{aligned}$$

Now we rewrite these:

$$\begin{aligned} x_1 x_r^{-1} x_r x_1 x_2 x_r^{-1} x_r x_2 \cdots x_r x_r^{-1} x_r x_r &= 1 \\ x_r x_1 x_1 x_r^{-1} x_r x_2 x_2 x_r^{-1} \cdots x_r x_r x_r x_r^{-1} &= 1. \end{aligned}$$

Finally, we rewrite these in the generators for  $H$ :

$$\begin{aligned} y_1 z_1 \cdots y_{r-1} z_{r-1} z_r &= 1 \\ z_1 y_1 \cdots z_{r-1} y_{r-1} z_r &= 1 \end{aligned}$$

An easy Tietze transformation eliminates  $z_r$ , and we have:

$$H = \langle y_1, \dots, y_{r-1}, z_1, \dots, z_{r-1} \mid y_1 z_1 \cdots y_{r-1} z_{r-1} = z_1 y_1 \cdots z_{r-1} y_{r-1} \rangle.$$

**Exercise A.15.** Prove (A.13).

**Definition A.16.** A group is finitely presented if it has at least one finite presentation, i.e., a presentation that employs only finitely many generators and finitely many relations.

**Observation A.17.** *It follows immediately from (A.13) that a subgroup of finite index in a finitely presented group is finitely presented.*

**Exercise A.18.** Prove that a virtually finitely presented group is finitely presented.

**Exercise A.19.** Let  $G$  be finitely generated. Show that every generating set for  $G$  contains a finite subset that already generates  $G$ .

**Exercise A.20.** Let  $G$  be finitely presented. Show that any finite generating set for  $G$  can be extended to a finite presentation for  $G$ .

**Exercise A.21.** Let

$$G = \langle x_1, \dots, x_r \mid r_1, r_2, \dots \rangle$$

be a finitely presented group. Show that in the above presentation all but finitely many relations are redundant.

**Exercise A.22.** Prove that finitely presented by finitely presented groups are finitely presented.

In passing from the Cayley graph  $\Gamma_\Sigma(G)$  to the Cayley complex  $\Gamma_{\Sigma,R}(G)$  we killed the fundamental group by gluing in 2-cells. This might introduce non-trivial  $\pi_2$ . Of course, we could go on and kill this  $\pi_2$  by gluing in 3-cells in a  $G$ -equivariant way, at the cost of, maybe, introducing non-trivial  $\pi_3$ . We can continue and kill all homotopy groups. We get a contractible free  $G$ -complex. Given this construction, what will the presentation 2-complex turn into? The answer is given in the following definition.

**Definition A.23.** An Eilenberg-Maclane-complex for  $G$  is a CW-complex with fundamental group  $G$  and contractible universal cover.

Confer section 2.4.1, in particular (2.50) and (2.51), for more on Eilenberg-Maclane spaces and their relation to group cohomology.

## A.2 Constructions

Here is a brief overview on various way of building groups on top of other groups.

### A.2.1 Direct, Semidirect, and Wreath Products

**Definition A.24.** Let  $(G_i)_{i \in I}$  be a family of groups. The cross product of the  $G_i$  is the group

$$\times_{i \in I} G_i := \text{Maps}(I, G)$$

where multiplication is defined pointwise.

The direct product of the  $G_i$  is the subgroup of the product consisting of those families  $(g_i)$  where  $g_i = 1$  for all but at most finitely many  $i \in I$ . We denote the direct product by

$$\prod_{i \in I} G_i.$$

Note that for finite index sets  $I$ , the notion of cross product and direct product coincides.

**Definition A.25.** A homomorphism  $\pi : G \rightarrow Q$  is a retraction if there is a homomorphism  $\sigma : Q \rightarrow G$  such that the composition  $Q \xrightarrow{\sigma} G \xrightarrow{\pi} Q$  is the identity. Note that in this case  $\pi$  has to be surjective. The homomorphism  $\sigma$  is called a (group theoretic) section.

A short exact sequence of groups is a sequence of groups and homomorphisms

$$N \xrightarrow{\iota} G \xrightarrow{\pi} Q$$