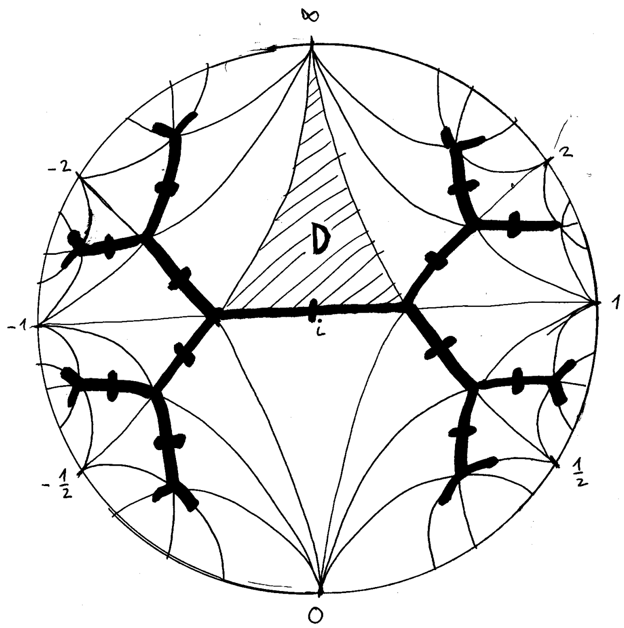
5.2.3 The Tree of $\mathrm{SL}_2(\mathbb{Z})$

Let us draw the tiling (5.50) in the unit disc model:



We see that the finite edges of the tiling triangles form a trivalent tree.

Proposition 5.56. *We may trust our eyes, the finite edges of the tiling triangles do, indeed, form a tree.*

Proof. There is no doubt that these edges form a graph. We have to argue that this graph is contractible. Thus it suffices to recognize this graph a a deformation retract of $\mathbb{H}^2$.

In the fundamental triangle $\overline{D}$, we define a retraction by moving every point x down along the geodesic from ∞, the ideal triangle vertex, through x at hyperbolic unit speed until the point hits the bottom boundary edge of $\overline{D}$. This retraction, being defined entirely in terms of the hyperbolic metric and the geometry of $\overline{D}$, obviously extends $\mathrm{SL}_2(\mathbb{Z})$-equivariantly to all of $\mathbb{H}^2$. **q.e.d.**

Definition 5.57. A group G acts properly discontinuously on a topological space X if, for every compact subset $C \subseteq X$, the set

$$\{g \in G \mid gC \cap C \neq \emptyset\}$$

is finite.

Corollary 5.58. $\mathrm{SL}_2(\mathbb{Z})$ *acts cocompactly and properly discontinously on a tree.*

Proof. Cocompactness is obvious. For the compact subset given by the fundamental edge e, we already established that the action is properly discontinuous. The general case follows: Let C be a compact subset of the tree. Then it is covered by finitely many edges $M_i e$. Thus,

$$MC \cap C \neq \emptyset$$

implies

$$MM_i e \cap M_j e \neq \emptyset$$

which in turn yields

$$M = M_j x M_i^{-1}$$

for some $x \in \Sigma$. It is apparent that only finitely many elements arise that way. **q.e.d.**

Corollary 5.59. $\mathrm{SL}_2(\mathbb{Z})$ *is of type* F_∞.

Proof. This follows from the action on the tree in view of (C.10). **q.e.d.**

Exercise 5.60. Show that $\mathrm{SL}_2(\mathbb{Z})$ contains a non-abelian free subgroup of finite index.

Exercise 5.61. Let G act on a tree T such that the following conditions are satisfied:

1. G acts transitively on the set of geometric (unoriented) edges.

2. G does not act transitively on the set of vertices.

3. T has no terminal vertices and is not isomorphic to a line.

Let e be an edge in T that connects the vertices v and w. Let G_v and G_w denote the stabilizers of these vertices and let G_e denote the stabilizer of the edge e. Then $G = G_v *_{G_e} G_w$.

Infer that

$$\mathrm{SL}_2(\mathbb{Z}) = C_6 *_{C_2} C_4.$$

5.2.4 The Conjugacy Problem

Definition 5.62. A group G is combable with respect to the generating set Σ, if there is a constant C and distinguished paths from 1_G to each vertex $v \in \Gamma_\Sigma(G)$ that have the C-fellow traveler property. That is, whenever we have two distinguished paths p and q starting at $1_G \in \Gamma_\Sigma(G)$ whose endpoints have distance ≤ 1, the following inequality holds along the paths:

$$d(p_t, q_t) \leq C. \tag{8}$$

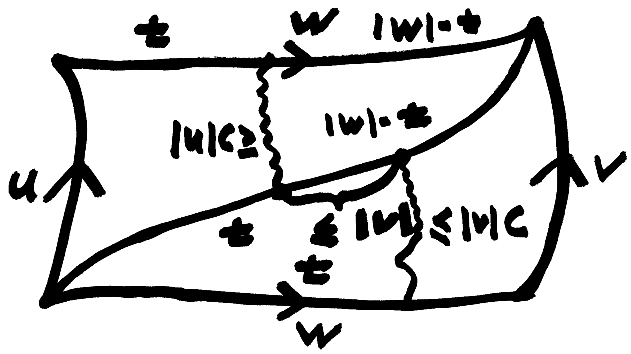
Here the paths are traversed with unit speed. If all the combing paths can be chosen to be geodesics, the group G is called geodesically combable. If any two geodesic paths whose endpoints have distance ≤ 1 satisfy the inequality (8), we say that G has the C-fellow traveler property.

Exercise 5.63. Prove that for some finite generating set and some constant C, geodesic paths in the Cayley graph of $\mathrm{SL}_2(\mathbb{Z})$ have the C-fellow traveler property. Hint: Use geodesic paths and the action on the tree.

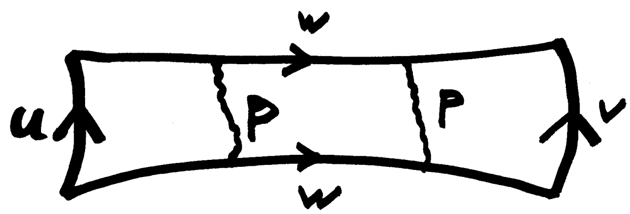
Proposition 5.64. *Combable groups have solvable conjugacy problem.*

Proof. Let u and v be two words. We show that we can restrict the search for a conjugating element w with $wu = vw$ to words of a length bounded by a constant that depends only on $|u|$ and $|v|$.

So let w be a shortest conjugating element – if there is one. We have to give an upper bound for $|w|$. This is done in two pictures:



This picture proves that the upper and lower geodesic are fellow travelers for a constant essentially proportional to $|u| + |v|$.



This picture shall remind you of the pigeon hole principle. If w is too long, then there will be two points such that the short vertical geodesic connections read identical group elements. We can then cut out the middle rectangle and shorten the conjugation rectangle. **q.e.d.**

5.2.5 Finite Quotients and Congruence Subgroups

We note an immediate consequence of (5.61).

Observation 5.65. *$C_3 * C_2$ is a quotient of $\mathrm{SL}_2(\mathbb{Z})$. In fact, it follows from (5.61) that $C_3 * C_2 = \mathrm{PSL}_2(\mathbb{Z})$.* **q.e.d.**

It follows that every group that can be generated by two elements of orders two and three is a quotient of $\mathrm{SL}_2(\mathbb{Z})$.

Exercise 5.66. Show that the alternating group A_{11} is a quotient of $\mathrm{SL}_2(\mathbb{Z})$.