Proof. Since $F$ is infinite, any finite quotient is proper whence abelian. Therefore, the elements of the commutator subgroup cannot be separated from the identity in any finite quotient. The commutator subgroup, however, is non-trivial — in fact, it is infinite. \hfill q.e.d.

Exercise 4.42. Show that $F$ is torsion-free.

Theorem 4.43. Every non-abelian subgroup of $F$ contains a copy of $\mathbb{Z} \times \mathbb{Z}$.

Proof. Let $f$ and $g$ be two elements that do not commute. We have to find a copy of $\mathbb{Z} \times \mathbb{Z}$ in the subgroup $\langle f, g \rangle$.

Since $f$ has only finitely many break points, the closed set

$$ \{ t \mid f(t) = t \} $$

decomposes into finitely many intervals. As the same holds for $g$, it follows that the open set

$$ \{ t \mid f(t) \neq t \text{ or } g(t) \neq t \} $$

decomposes into finitely many open intervals $I_0, \ldots, I_r$.

We consider the commutator $f_0 := [f, g]$. We will find a conjugate of $f_0$ inside $\langle f, g \rangle$ that commutes with $f_0$. The key idea is to conjugate such that the support of $f_0$ in $I_0$ is moved off itself, which is possible since this homomorphism $f_0$ is the identity in small neighborhoods of the endpoints of $I_0$. So we only have to prove the existence of this conjugating element.

Let $t$ be any point in $I_i$. Obviously, $\inf \langle f, g \rangle \cdot t$ is a global fix point for the action of $\langle f, g \rangle$ in the closure of $I_i$. Clearly, it has to be $\inf I_i$. Hence this point is a limit point of the orbit of $t$. Hence the problem in the preceding paragraph has a solution and we can conjugate $f_0$ as to move its support off itself inside $I_0$. Call this conjugate $g_0$.  

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Now, consider \( f_1 := [f_0, g_0] \). By construction, this element is the identity on \( I_0 \). So we run the same argument as above in the interval \( I_1 \). This way, we can take care of all intervals. Eventually, we have two elements \( f_r \) and \( g_r \) that commute. \textit{q.e.d.}

**Corollary 4.44.** \( F \) does not contain non-abelian free groups. \textit{q.e.d.}

### 4.3 Amenability

It is not known whether \( F \) is amenable. However, if \( F \) is amenable, it is at least not obviously so.

**Theorem 4.45.** Thompson’s group \( F \) is not elementary amenable.

**Proof.** We prove \( F \not\in EG_\alpha \) for all \( \alpha \). And, of course, this is done by transfinite induction or contradiction. We proceed by contradiction. So assume there was an ordinal \( \alpha \) with \( F \in EG_\alpha \). Then we can take \( \alpha \) to be minimal with this property.

Since \( F \) is neither abelian nor finite, \( \alpha > 0 \) and so \( F \) sneaked into \( EG_\alpha \) as an extension or as a direct union of groups in lower strata. The direct union possibility is ruled out since \( F \) is finitely generated. Thus, we have to deal with ways of representing \( F \) as an extension.

Let \( N_\beta \hookrightarrow F \twoheadrightarrow Q_0 \) be a short exact sequence where \( Q_0 \) is a proper quotient. By (4.34), \( Q_0 \) is abelian. Hence \( N_\beta \in EG_\beta \) contains \( [F,F] \). Hence \( [F,F] \in EG_\beta \). However, since \( [F,F] \) contains a copy of \( F \), it follows that \( F \in EG_\beta \). Thus, \( \alpha \) was not minimal. \textit{q.e.d.}

### 4.4 Finiteness Properties

We will construct a very nice (i.e., contractible) cube complex upon which \( F \) acts. The construction is due to K.S. Brown and R. Geoghegan [BrGe84].

This construction starts from the Cayley complex for the infinite presentation
\[
\tilde{F} := \langle x_0, x_1, \ldots \mid x_m x_i = x_i x_{m+1} \text{ for } i < m \rangle.\
\]
Thus, we have a vertex for each element in $F$. From each vertex countably many edges issue, and the relations give rise to squares. This complex $\Gamma_F$ is simply connected. Our goal is to fill in cubes as to kill off higher homotopy groups. To do this, we fill in all the obvious cubes: For any square whose edges are labeled by $j, k$ and $k+1$ – we assume $j < k$ –, and any $i < j$, we have a cube

![Diagram](image)

We continue in higher dimensions. Thus for each vertex in $\Gamma_F$ and each tuple $(q_0 < q_2 < \ldots < q_r)$, we have a cube issuing from that vertex whose edges are labeled by the indices $q_i$. Note that each of these cubes has a unique vertex with all edges pointing away (the source) and a unique vertex with all edges coming in (the sink). Let $Y$ denote the cube complex constructed this way, and put $X := F \setminus Y$.

**Theorem 4.46 ([BrGe84]).** $Y$ is contractible.

**Proof.** Let $P$ be the monoid of positive elements in $F$ and let $Y_+$ be the subcomplex of $Y$ spanned by $P \subseteq Y$. This is the complex formed by all those cubes whose sink is in $P$.

**Claim A.** $Y_+$ is contractible.
Proof. We will do Morse theory. There is an obvious height function on the vertices: the length of the positive word — or alternatively, the number of non-leaf nodes in the forest representing that vertex. Moving along an outgoing edge increases the height by 1. Thus, the height extends linearly to the cubes. This is a combinatorial Morse function as it is non-constant on edges. Since the minimal set is just one point (the identity vertex), contractibility of $Y_+$ would follow from contractibility of descending links ( ).

The descending link is the part of the link in a vertex $v$ spanned by its incoming edges. These correspond to splitting off a generator on the right, i.e., deleting a terminal caret in the forest. Thus we can tell which incoming edges issue from vertices in $Y_+$. More importantly, we see that we can delete all terminal carets and find a source in $Y_+$ which gives rise to a simplex in the link of $v$ that connects all the descending vertices. Thus, we proved that the descending link is a simplex and therefore contractible. The claim follows. 

Now we consider the cover of $Y$ by translates $fY_+$.

Claim B. For any finite set of elements $f_0, \ldots, f_m \in F$, there is an element $f \in F$ such that

$$f_0Y_+ \cap \cdots \cap f_mY_+ = fY_+.$$ 

Proof. This follows from the corresponding claim about translates $fP$ which in turn follows by induction from the claim that for any $f \in F$, there is a $g \in P$ such that

$$P \cap fP = gP.$$ 

Writing $f$ as a pair of trees, we see that in order to find a positive right multiple of $f$, we first have to annihilate the bottom tree. Afterward, we will end up with a right multiple of

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the top tree. Thus, $g$ is the element represented by the top tree of $f$. □

From this claim, it follows at once, that the translates $fY_+$ form a cover of $Y$ by contractible subcomplexes such that any intersection of these subcomplexes is contractible. The nerve of the cover is the simplex with vertex set $F$. Hence, by (B.11), the space $Y$ has the homotopy type of a big simplex. \textbf{q.e.d.}

\textbf{Theorem 4.47.} $Y$ is CAT(0).

\textbf{Proof.} The edges around each vertex $v$ come in pairs: For each generator $x_i$, there is one incoming edge and one outgoing edge, both labeled by $i$. We denote the incoming edge by the pair $(i,1)$ and the outgoing edge by $(i,1)$. These pairs represent the vertices in $\text{lk}(v)$. In order to determine the simplices, we have to know which of these edges span a cube. Obviously, there are no cubes that involve both, $(i,0)$ and $(i,1)$. Hence we can represent cubes in the star of $v$ by tupels

$$((q_1, \varepsilon_1), \ldots, (q_1, \varepsilon_1))$$

with $q_i < q_{i+1}$.

The following picture assumes $i < j$ and shows the edge labels of all possible squares involving these to labels.
The incoming edges only can form a square if \( i < j - 1 \). In general, a tupel

\[
((q_1, \varepsilon_1), \ldots, (q_1, \varepsilon_1))
\]
of edges spans a cube if \( q_i < q_{i+1} - \varepsilon_i \).

This condition, however, is satisfied for a tupel if and only if it is satisfied for all of its subtupels of size two. This is, a set of vertices in \( \text{Lk}(v) \) spans a simplex if and only if any two of them are joined by an edge. Thus, \( \text{Lk}(v) \) is a flag complex. It follows from Gromov's lemma that \( Y \) is \( \text{CAT}(0) \) provided it is simply connected. That, in turn follows from the fact that the 2-skeleton of \( Y \) is a Cayley complex for \( F \). \( \text{q.e.d.} \)

We give a description of the quotient \( X \). This is also a cube complex. \( X \) has one vertex. The edges in the 1-skeleton are indexed by non-negative integers. We index higher dimensional cubes by the indices of the edges pointing toward the sink of the cube. Then, an ascending sequence

\[
\mathbf{q} := (q_1, \ldots, q_m)
\]

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describes a cube if and only if $0 \leq F_1$ and $q_i + 2 \leq q_{i+1}$. The $2m$ faces of the cube $q$ are given by the boundary operators

$$
\partial_i^+ q := (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_m)
$$

$$
\partial_i^- q := (q_1, \ldots, q_{i-1}, q_{i+1} - 1, \ldots, q_{m-1}).
$$

**Theorem 4.48.** $X$ is homotopy equivalent to a CW-complex that has one vertex and two cells in each dimension $\geq 1$.

**Proof.** We distinguish three types of cells. For each dimension $\geq 1$, we have two essential cells $(0,3, \ldots, 3m-3)$ and $(1,4, \ldots, 3m-2)$. We call an $m$-cube $q = (q_1, \ldots, q_m)$ collapsible if there is an index $i$ such that $q_{i+1} \neq q_i + 3$ and such that, for the last index $j$ with that property, $q_{j+1} = q_j + 2$. In this case

$$
\partial_j^+ q = (q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_m)
$$

is called the free face of the collapsible cube $q$. Cells that are neither essential nor collapsible are considered redundant.

Observe that the free face $\partial_j^+ q$ of a collapsible cube $q$ is redundant since $q_{j+1} \geq q_{j-1} + 4$ and $q_{i+1} = q_i + 3$ for all $i > j + 1$.

On the other hand, each redundant cell $p = (p_1, \ldots, p_m)$ is the free face of a unique collapsible cell since for the last index $i$ with $p_{i+1} \neq p_i + 3$, we have $p_{i+1} \geq p_i + 4$ whence

$$(p_1, \ldots, p_i, p_{i+1} - 2, p_{i+2}, \ldots, p_m)$$

is a collapsible cell with free face $p$ – note that we have to allow $i = 0$ here because of cubes like $(2,5,8,\ldots)$.

Now consider the collapsible cube $q = (q_1, \ldots, q_m)$ with free face $\partial_j^+ q$ which is to say that we have

$$q_i = q_j + 3(i - j) - 1$$

for all $i > j$. The following statements are easy to see:

1. For $i < j$, the face $\partial_i^- q$ is collapsible.

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2. For $i > j$, the tuple $\partial_i^i \mathbf{q}$ precedes the free face $\partial_j^j \mathbf{q}$ in the lexicographic order.

3. Also, $\partial_j^j \mathbf{q}$ precedes the free face in the lexicographic order.

We define $X_{-0} := X_{+0} := X_0$, and inductively

\[ X_{-m} := X_{+m-1} \cup \text{essential cells in dimension } m \]
\[ X_{+m} := X_{-m} \cup \text{redundant cells in dimension } m \cup \text{collapsible cells in dimension } m \cdot \]

Obviously, we have

\[ X_{+m-1} \subseteq X_{-m} \subseteq X_m \subseteq X_{+m} \cdot \]

Moreover, we have $X_{-m} \simeq X_{+m}$ since the latter is obtained from the former by a transfinite sequence of elementary expansions which are performed according to the lexicographic order. Since the lexicographic order is a well ordering, this actually makes sense.

Now, we are ready to construct the space $Z$ which is homotopy equivalent to $X$ but uses at most two cells per dimension. Put $Z_0 := X_0$. Higher skeleta are constructed by induction together with the homotopy equivalence. So assume we have already a homotopy equivalence

\[ \pi_{m-1} : X_{+m-1} \to Z_{m-1} \cdot \]

Put

\[ Z_m := Z \cup B_0(m) \cup B_1(m) \]

where the attaching maps are induced by the maps that embed the boundaries of the two essential $m$-cubes in $X_{+m-1}$. Thus we obtain a homotopy equivalence

\[ \pi_m : X_{-m} \to Z_m \]

and $X_{-m} \simeq X_{+m}$ gives us the equivalence

\[ \pi_m : X_{+m} \to Z_m \]

which completes the proof. \hspace{1cm} \textbf{q.e.d.}

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