4 Thompson's Groups

Definition 4.1. Thompson's Group F is defined by the following infinite presentation:

$$F := \langle x_0, x_1, \ldots \mid x_m x_i = x_i x_{m+1} \texttt{ for } i < m \rangle .$$

Exercise 4.2. Show that the following presentations also define Thompson's Group F:

$$\begin{aligned} F_{\texttt{finite,a}} &:= \left\langle a, b \,\middle|\, b^{aa} = b^{ab},\, b^{abb} = b^{abb^a} \right\rangle \\ F_{\texttt{finite,b}} &:= \left\langle c, d \,\middle|\, d^{cc} = d^{cd},\, d^{cdc} = d^{cdd} \right\rangle . \end{aligned}$$

Here, we use the convention $g^h := h^{-1} g h$.

Corollary 4.3. *Thompson's Group F is finitely presented.*

Observation 4.4. *For $i < m$, the following relations hold in F:*

$$\begin{aligned} x_m x_i &= x_i x_{m+1} \\ x_i^{-1} x_m^{-1} &= x_{m+1}^{-1} x_i^{-1} \\ x_m^{-1} x_i &= x_i x_{m+1}^{-1} \\ x_i^{-1} x_m &= x_{m+1} x_i^{-1} . \end{aligned}$$

Hence every element $f \in F$ can be written as a word in normal form:

$$f = x_0^{a_0} \cdots x_r^{a_r} x_r^{-b_r} \cdots x_0^{-b_0}$$

where

1. $a_i, b_i \geq 0$.
2. $0 \in \{a_r, b_r\} \neq \{0\}$.
3. *If* $a_i > 0 < b_i$, *then* $\{a_r, b_r\} \neq \{0\}$.

Remark 4.5. In (4.28), we will show that this normal form is unique.

Proposition 4.6. *F is infinite. In fact, the abelianization is $F^{ab} = C_\infty \times C_\infty$.*

Proof. From the presentation, we have

$$\begin{aligned} F^{\mathrm{ab}} &= \langle x_0, x_1, \dots \mid x_i x_m = x_m x_i = x_i x_{m+1} \texttt{ for } i < m \rangle \\ &= \langle x_0, x_1, \dots \mid x_0 x_1 = x_1 x_0,\ x_i = x_{i+1} \texttt{ for } 1 \leq i \rangle \end{aligned}$$

which is a presentation of $C_\infty \times C_\infty$. **q.e.d.**

4.1 An Action of Thompson's Group F

Definition 4.7. Let $\mathbb{R}_{\geq 0}^{\infty} := [0, \infty)$ be the positive real numbers including infinity. The Group $F(\frac{1}{2}, \mathbb{R}_{\geq 0}^{\infty})$ is the group of piece wise linear self-homeomorphism f of $\mathbb{R}_{\geq 0}^{\infty}$ satisfying:

1. Slopes change only finitely many times.
2. Slopes changes only a places in $\mathbb{Z}\left[\frac{1}{2}\right]$.
3. Slopes are in $2^{\mathbb{Z}}$.
4. There is an integer s such that

   $$f^t = t + s$$

   for all sufficiently large t.

Remark 4.8. Since there is a piece wise linear homeomorphisms $\mathbb{I} \to \mathbb{R}_{\geq 0}^{\infty}$ identifying the unit interval an $\mathbb{R}_{\geq 0}^{\infty}$, we have an isomorphism

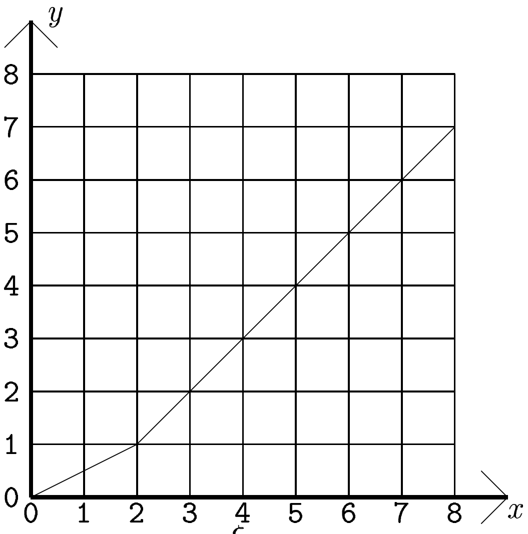
$$F(\frac{1}{2}, \mathbb{R}_{\geq 0}^{\infty}) = F(\frac{1}{2}, \mathbb{I}) := \mathrm{PL}_{\mathbb{Z}[\frac{1}{2}], 2^{\mathbb{Z}}}(\mathbb{I})$$

of $F(\frac{1}{2}, \mathbb{R}_{\geq 0}^{\infty})$ and the group $F(\frac{1}{2}, \mathbb{I})$ of piece wise linear self-homeomorphism of the unit-interval with finitely many break points all in $\mathbb{Z}\left[\frac{1}{2}\right]$ and slopes in $2^{\mathbb{Z}}$.

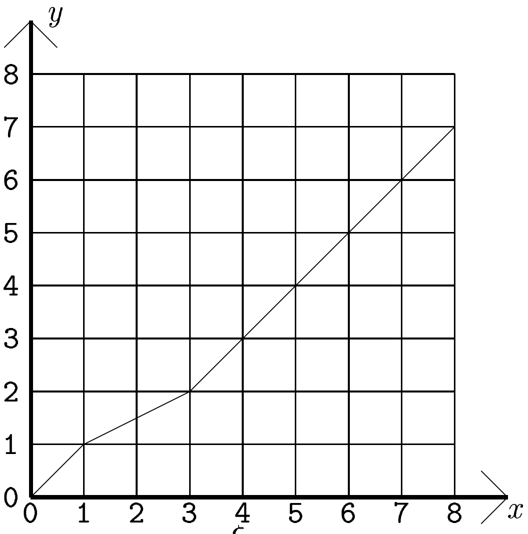
Note that $F(\frac{1}{2}, \mathbb{I})$ also is the strict subgroup of $F(\frac{1}{2}, \mathbb{R}_{\geq 0}^{\infty})$ of those homeomorphism that are the identity on $[1, \infty)$.

Let us have a closer look at those piece wise linear self-homeomorphisms of $\mathbb{R}_{\geq 0}^{\infty}$:

Example 4.9. The following maps are in $F(\frac{1}{2}, \mathbb{R}^{\infty}_{\geq 0})$:



ξ_0



ξ_1

In general, we define ξ_i to be the function that has slope 1 on $[0, i]$ and $[i + 2, \infty]$ and has slope $\frac{1}{2}$ in $[i, i + 2]$.

Exercise 4.10. Prove that the map

$$x_i \mapsto \xi_i$$

extends to a group homomorphism

$$F \to F(\frac{1}{2}, \mathbb{R}^{\infty}_{\geq 0}).$$

4.1.1 Forest Diagrams

Definition 4.11. A standard dyadic interval is an interval of the form $\left[\frac{n}{2^m}, \frac{n+1}{2^m}\right]$ where n and m are non-negative integers. Let us call a standard dyadic interval of length 1 a standard unit interval. Note that this implies that its boundary points are integer.

A finitary dyadic decomposition of $\mathbb{R}^{\infty}_{\geq 0}$ is a set $\{I_0, I_1, \ldots\}$ of standard dyadic intervals satisfying:

1. $\mathbb{R}^{\infty}_{\geq 0} = \biguplus_{i \geq 0} I_i$.

2. For all i sufficiently large, I_i has length 1.

Let $\mathcal{I}$ and $\mathcal{J}$ be two finitary dyadic decompositions of $\mathbb{R}^{\infty}_{\geq 0}$. We say that $\mathcal{J}$ is a <u>refinement</u> of $\mathcal{I}$ – and that $\mathcal{I}$ is <u>coarser</u> than $\mathcal{J}$ – if every interval in $\mathcal{J}$ is contained in an interval from $\mathcal{I}$.

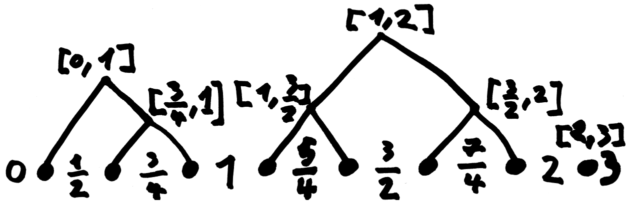
The <u>trivial finitary dyadic decomposition</u> is the set of standard unit intervals in $\mathbb{R}^{\infty}_{\geq 0}$. Every finitary dyadic decomposition is a refinement of the trivial one.

Observation 4.12. *Note that if you pick a standard dyadic interval in a finitary dyadic decomposition of* $\mathbb{R}^{\infty}_{\geq 0}$ *and replace it by its two halves, you obtain another finitary decomposition. We call this operation an <u>elementary split</u>. The result of a split is always a refinement. Vice versa, every refinement of a decomposition can be obtained by finitely many splits.*

Definition 4.13. An <u>infinite binary forest</u> is a directed tree where each vertex has two outgoing edges one of which is labeled as left whereas the other is labeled as right. The endpoints of these two edges are the <u>children</u> of the vertex. Each vertex is the <u>parent</u> of its children. A <u>finite binary forest</u> is a subgraph of an infinite binary forest all of whose vertices either have two children or none.

Example 4.14. The standard dyadic intervals form a forest $\mathcal{F}^{(2)}$. Each standard dyadic interval I is a vertex in $\mathcal{F}^{(2)}$ and its two children are the left and right half of I, respectively.

Observation 4.15. *Every finitary dyadic decomposition* $\mathcal{I}$ *of* $\mathbb{R}^{\infty}_{\geq 0}$ *defines a subforest* $\mathcal{F}^{(2)}_{\mathcal{I}} \subseteq \mathcal{F}^{(2)}$*: The intervals in* $\mathcal{I}$ *are vertices of* $\mathcal{F}^{(2)}$ *and* $\mathcal{F}^{(2)}_{\mathcal{I}}$ *is the minimal subforest containing all of them. Since a standard dyadic interval of length* 2^{-m} *corresponds to a vertex of distance* m *to a root vertex, you can actually read off the decomposition* $\mathcal{I}$ *from* $\mathcal{F}^{(2)}_{\mathcal{I}}$ *pretty easily by looking at the "spaces between the leafs" as indicated in this example:*



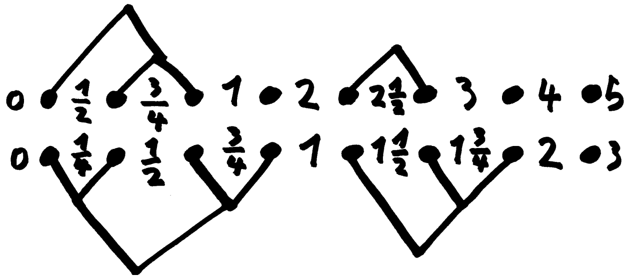
We are interested in finitary dyadic decompositions because they provide a convenient way of describing elements of $F(\frac{1}{2}, \mathbb{R}^{\infty}_{\geq 0})$. The reason is that if you map a standard dyadic interval linearly to another standard dyadic interval, the slope of the map is a power of 2. Hence a pair $(\mathcal{I}, \mathcal{J})$ of finitary dyadic decompositions describes an element $\varphi_{(\mathcal{I},\mathcal{J})} \in F(\frac{1}{2}, \mathbb{R}^{\infty}_{\geq 0})$ as follows: The left-most interval in $\mathcal{I}$ is mapped linearly to the left-most interval in $\mathcal{J}$, and from there you proceed from left to right matching intervals in $\mathcal{I}$ with intervals in $\mathcal{J}$.

Definition 4.16. A pair of finitary standard dyadic decompositions of $\mathbb{R}^{\infty}_{\geq 0}$ is called a forest diagram.

Observation 4.17. *If* $(\mathcal{I}, \mathcal{J})$ *is a forest diagram representing* φ*, then* $(\mathcal{J}, \mathcal{I})$ *is a forest diagram that represents* φ^{-1}*.*

Because of the way the intervals are supposed to match up, we draw the forest diagram $(\mathcal{I}, \mathcal{J})$ as a top forest with leafs pointing down and a bottom forest with leafs pointing up such that leafs corresponding to matching intervals are aligned.

Example 4.18. The diagram



represents the piece wise linear map that interpolates the chart

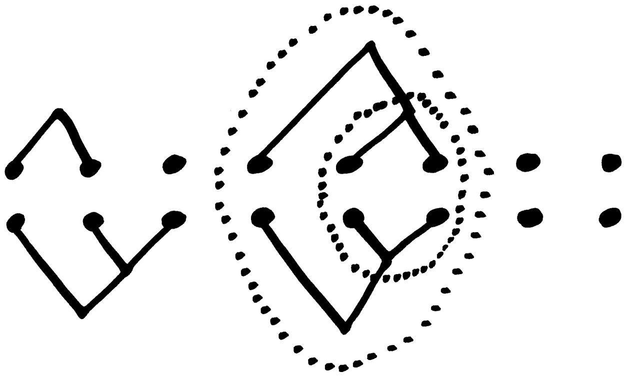
$$\begin{array}{rcl}
0 & \mapsto & 0 \\
\frac{1}{4} & \mapsto & \frac{1}{2} \\
\frac{1}{2} & \mapsto & \frac{3}{4} \\
\frac{3}{4} & \mapsto & 1 \\
1 & \mapsto & 2 \\
1\frac{1}{2} & \mapsto & 2\frac{1}{2} \\
1\frac{3}{4} & \mapsto & 3 \\
2 & \mapsto & 4
\end{array}$$

and has slope 1 thereafter.

There are, of course, different forest diagrams that define the same element of $F(\frac{1}{2}, \mathbb{R}^{\infty}_{\geq 0})$ – for instance, any symmetric forest diagram $(\mathcal{I}, \mathcal{I})$ defines the identity. More general, any forest diagram can be expanded without changing the homeomorphism it represents by simultaneously splitting matching top and bottom leafs. Conversely, deleting of matching symmetric subtrees will also not alter the homeomorphism.

Definition 4.19. A forest diagram that does not contain a matching pair of symmetric trees is called reduced.

Note that any non-reduced forest diagram contains a matching pair of carets:



The importance of forest diagrams derives from the following

Theorem 4.20. *Every element* $\varphi \in F(\frac{1}{2}, \mathbb{R}^{\infty}_{\geq 0})$ *is represented by a unique reduced forest diagram.*

Proof. Let I be a closed interval with dyadic rational endpoints such that φ is linear on I. Since the slope of $\varphi|_I$ is a power of 2, the image $\varphi(I)$ also has dyadic rational endpoints. Since a sufficiently high power of $\frac{1}{2}$ divides evenly into the endpoints of I and $\varphi(I)$, we can find a decomposition of I into standard dyadic intervals that maps to a decomposition of $\varphi(I)$ by standard dyadic intervals. Since φ has only finitely many breakpoints, existence of a forest diagram representing φ follows from the fact that for sufficiently large t, the map φ acts as an integer shift, which implies that "far to the right" we can describe φ by matching standard unit intervals.

Given existence of forest diagrams representing φ, we obtain a reduced one by, well, reducing any unreduced forest diagram for φ.

As for uniqueness, consider the maximum standard intervals that have the following properties:

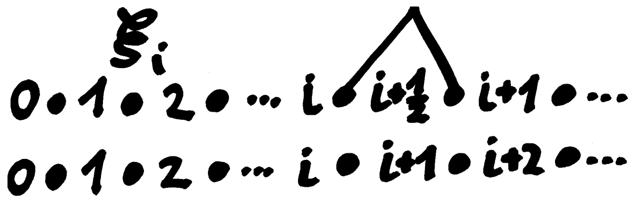
1. The homeomorphism φ restricts to a linear map.
2. The image of the interval under φ is a standard dyadic interval.

Observe that these intervals form a finitary dyadic decomposition $\mathcal{I} := \mathcal{I}_\varphi$ of $\mathbb{R}^{\infty}_{\geq 0}$ which in turn defines a subforest $\mathcal{F}^{(2)}_{\varphi}$ of $\mathcal{F}^{(2)}$ that is contained in the bottom forest of any forest diagram for φ. On the other hand, it is obvious from the definition that φ takes the decomposition $\mathcal{I}$ to a finitary dyadic decomposition $\mathcal{J}$. It follows that $(\mathcal{I}, \mathcal{J})$ is a reduced forest diagram for φ to which any other forest diagram for φ reduces. **q.e.d.**

4.1.2 Normal Forms

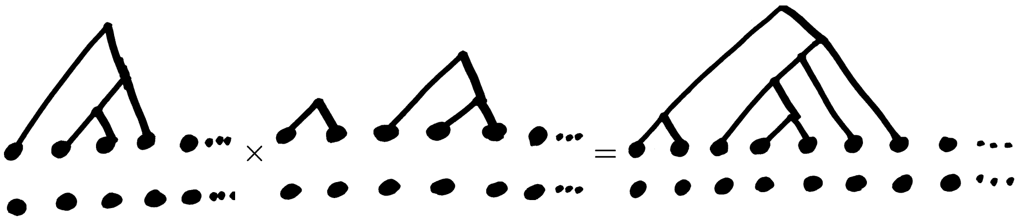
We have yet to prove that normal forms are unique. Eventually, this will follow from (4.20), but we have to exhibit the interplay between reduced forest diagrams and normal forms first.

Example 4.21. The elements ξ_i is represented by the following forest diagram:



Definition 4.22. An element $\varphi \in F(\frac{1}{2}, \mathbb{R}^{\infty}_{\geq 0})$ is called <u>positive</u> if its reduced forest diagram has a depth 0 bottom forest, i.e., the homeomorphism φ takes every standard unit interval standard dyadic interval.

Observation 4.23. *Positive elements in $F(\frac{1}{2}, \mathbb{R}^{\infty}_{\geq 0})$ form a submonoid, i.e., the product of two positive elements is positive. In fact, it is easy to multiply the corresponding forest diagrams. Let φ and ψ be two positive elements with top forests $\mathcal{I}_\varphi$ and $\mathcal{I}_\psi$, respectively. As the roots of $\mathcal{I}_\psi$ correspond to standard unit intervals, they match with the vertices of the bottom forest for φ. Hence the top tree for the product $\varphi\psi$ is obtained from $\mathcal{I}_\varphi$ and $\mathcal{I}_\psi$ by identifying the leafs of $\mathcal{I}_\varphi$ with the roots of $\mathcal{I}_\psi$. Here is an example:*



To put this as a slogan: In multiplication of positive element, you put the left factor on top of the right factor. An immediate consequence is the

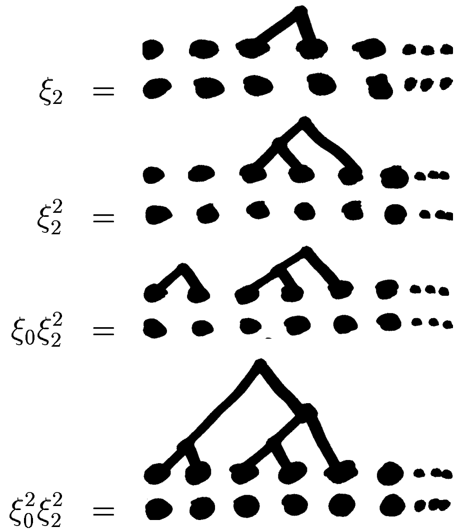
Proposition 4.24. *The reduced forest diagram for the element*

$$\xi_0^{a_0} \cdots \xi_r^{a_r}$$

has trivial bottom forest and its top forest is given by the following criterion:

> *The leaf at position* i *has a right-ascending edge path of length* a *but not of length* $a+1$.

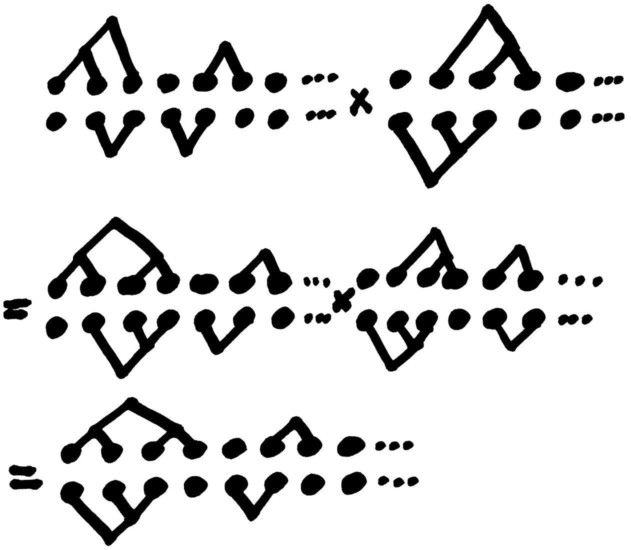
Proof. First, let us illustrate the criterion by one example:



This already is a proof: The criterion holds for the elements ξ_i and since we are multiplying from the left by elements whose indices do not exceed the indices that have been dealt with, we do not mess up the criterion in each step. **q.e.d.**

Remark 4.25. Note that every forest is uniquely determined by the numbers a_i which give the maximum length of a right ascending edge path starting at leaf i. Conversely, any sequence of non-negative integers with finite support determines a forest. In particular, the set of positive elements is the submonoid generated by the ξ_i.

Observation 4.26. *In general, we have to think of a forest diagram as a quotient of two positive elements. Since we are not in an* *abelian setting, we cannot easily multiply quotient. The rule for multiplying two forest diagrams, therefore, is to unreduce both by splitting top an bottom leafs as to make the bottom forest of the left factor equal to the top forest of the of the right factor. Once this is done, the isomorphic forest cancel:*



Corollary 4.27. *The homomorphism* $F \to F(\frac{1}{2}, \mathbb{R}^{\infty}_{\geq 0})$ *is surjective.*

Theorem 4.28. *The normal forms of (4.4) are unique.*

Proof. First, recall that a normal form is a word of the form

$$x_0^{a_0} \cdots x_r^{a_r} x_r^{-b_r} \cdots x_0^{-b_0} = \left(x_0^{a_0} \cdots x_r^{a_r}\right) \left(x_0^{b_0} \cdots x_r^{b_r}\right)^{-1}$$

where

1. $a_i, b_i \geq 0$.
2. $0 \in \{a_r, b_r\} \neq \{0\}$.
3. If $a_i > 0 < b_i$, then $\{a_r, b_r\} \neq \{0\}$.

Condition (1) implies that both parts of the word give rise to a forest. Hence a normal form defines a forest diagram in an obvious way. The other two restrictions imply that this forest diagram is reduced. Since a reduced diagram is uniquely determined by the homeomorphism it defines (4.20), the claim follows. **q.e.d.**

This theorem has many consequences.

Corollary 4.29. *Thompson's group F has solvable word problem.* **q.e.d.**

Corollary 4.30. *Thompson's group F has exponential growth.*

Proof. x_0^{-1} and x_1 generate a free monoid inside F. To see this, write a word, say

$$x_0^{-1}x_0^{-1}x_0^{-1}x_1x_1x_0^{-1}x_0^{-1}x_1x_0^{-1}x_0^{-1}x_1x_1$$

and pull all the x_0^{-1} to the right as to obtain the normal form:

$$x_4x_4x_6x_8x_8x_0^{-7}.$$

As the subscripts remember how many letters x_0^{-1} passed by, two different words in the monoid have different normal forms and, therefore, represent different elements of F. **q.e.d.**

Corollary 4.31. *F has trivial center.*

Proof. Let $f = x_0^{a_0} \cdots x_r^{a_r} x_r^{-b_r} \cdots x_0^{-b_0}$ be a non-trivial element of F. We will show that it does not commute simultaneously with x_0 and x_1.

$\underline{a_0 > 0 < b_0}$: In this case, $x_0^{-1} f x_0$ is in normal form and visibly not equal to f.

$\underline{r > 0 \text{ and } b_0 = 0}$: Now, we have

$$\begin{aligned} x_0^{-1} f x_0 &= x_0^{-1} x_0^{a_0} x_1^{a_1} \cdots x_r^{a_r} x_r^{-b_r} \cdots x_1^{-b_1} \\ &= x_0^{a_0} x_2^{a_1} \cdots x_{r+1}^{a_r} x_{r+1}^{-b_r} \cdots x_2^{-b_1} \\ &\neq f. \end{aligned}$$

$\underline{r > 0 \text{ and } a_0 = 0}$: This is dealt with like the previous case.

$\underline{r = 0}$: Now, f is a power of x_0 and, therefore, does not commute with x_1. **q.e.d.**

Exercise 4.32. Show that the centralizer of x_1 in F is isomorphic to $F \times C_\infty$.

Exercise 4.33. Does the group

$$\langle \ldots x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots \mid x_m x_i = x_i x_{m+1} \text{ for } i < m \rangle$$

embed into F?

4.1.3 A Remark on Tree Diagrams

As noted in (4.8), we have an isomorphism

$$F(\frac{1}{2}, \mathbb{R}^{\infty}_{\geq 0}) = F(\frac{1}{2}, \mathbb{I})$$

whence we can identify the three groups F, $F(\frac{1}{2}, \mathbb{R}^{\infty}_{\geq 0})$, and $F(\frac{1}{2}, \mathbb{I})$. The notion of forest diagram carries over to $F(\frac{1}{2}, \mathbb{I})$ as this group can be viewed as a the subgroup of $F(\frac{1}{2}, \mathbb{R}^{\infty}_{\geq 0})$ whose elements have support in $[0,1]$, i.e, these homeomorphism are the identity outside $\mathbb{I}$. In fact, since $\mathbb{I}$ corresponds to a root in $\mathcal{F}^{(2)}$, the forest diagrams for elements in $F(\frac{1}{2}, \mathbb{I})$ are somewhat degenerate in that the top an bottom forest are almost trivial – only the first subtree can be non-trivial. Hence, when using $F(\frac{1}{2}, \mathbb{I})$ as a model for F, one often uses tree diagrams instead of forest diagrams. There is, however, not much of a difference.

4.2 Subgroups and Quotients

We already saw (4.31) that F has trivial center. In this section we will prove the

Theorem 4.34. *The group F has a simple commutator subgroup, and that all quotients of F are abelian.*

These two claims are strongly related as shown by the following simple

Lemma 4.35. *If a group G has trivial center and a simple commutator subgroup $[G,G]$, then every proper quotient of G is abelian.*

Proof. Let $N \trianglelefteq G$ be a non-trivial normal subgroup, and let $n \in N - \{1\}$ be a non-trivial element. As G has trivial center, there is an element $g \in G$ that does not commute with n. Hence $[n,g] \in [G,G] \cap N$ proves that $N \cap [G,G]$ is non-trivial. This is obviously a normal subgroup of G and hence a normal subgroup of $[G,G]$. As the latter group is simple, we have $N \cap [G,G] = [G,G]$ whence $[G,G] \leq N$. Hence G/N is abelian. **q.e.d.**

Thus, we only have to show that the commutator subgroup $[F,F]$ is simple. For this reason, we shall examine the commutator subgroup in various models. We start with the presentation.

Proposition 4.36. *An element* $f = x_0^{a_0} \cdots x_r^{a_r} x_r^{-b_r} \cdots x_0^{-b_0}$ *is in the commutator subgroup if and only if*

$$a_0 - b_0 = 0 = a_0 + \cdots + a_r - b_r - \cdots - b_0.$$

Proof. The canonical projection $F \to F^{\mathrm{ab}} = \mathbb{Z} \times \mathbb{Z}$ sends x_0 to $(1,0)$ and x_i to $(0,1)$ for $i \geq 1$. Hence the claim follows from the fact that

$$a_0 - b_0 = 0 = a_0 + \cdots + a_r - b_r - \cdots - b_0$$

is equivalent to

$$a_0 - b_0 = 0 = a_1 + \cdots + a_r - b_r - \cdots - b_1$$

which is easily seen. **q.e.d.**

Proposition 4.37.

This characterization translates into the other models for F straightforwardly.

Corollary 4.38. *A homeomorphism* $\varphi \in F(\frac{1}{2}, \mathbb{R}_{\geq 0}^{\infty})$ *is in the commutator subgroup if and only if it is the identity close to* 0 *and close to* ∞.

Proof. Assume $\varphi = \xi_0^{a_0} \cdots \xi_r^{a_r} \xi_r^{-b_r} \cdots \xi_0^{-b_0}$. The slope at 0 translates into $a_0 - b_0$ whereas $a_0 + \cdots + a_r - b_r - \cdots - b_0$ is the shift realized by φ for large arguments. **q.e.d.**

The same routine translation from one model to the next gives:

Corollary 4.39. *A homeomorphism $\varphi \in F(\frac{1}{2}, \mathbb{I})$ is in the commutator subgroup if and only if it is the identity close to 0 and close to 1.* **q.e.d.**

Lemma 4.40. *Let $f = x_0^{a_0} \cdots x_r^{a_r} x_r^{-b_r} \cdots x_0^{-b_0}$ be an element of F with $s := a_0 + \cdots + a_r - b_r - \cdots - b_0 > 0$. Then the normal closure of f contains the commutator subgroup $[F, F]$.*

Proof. Let N be the normal closure of f. Then, for sufficiently large N, we have

$$\begin{aligned} f^{-1} x_N f &= \left(x_0^{a_0} \cdots x_r^{a_r} x_r^{-b_r} \cdots x_0^{-b_0}\right)^{-1} x_N x_0^{a_0} \cdots x_r^{a_r} x_r^{-b_r} \cdots x_0^{-b_0} \\ &= x_{N + a_0 + \cdots + a_r - b_r - \cdots - b_0} \\ &= x_{N+s}. \end{aligned}$$

Hence we have

$$x_N \cong x_{N+s} \mod N.$$

From this, we infer first

$$x_1 \cong x_{1+s} \mod N$$

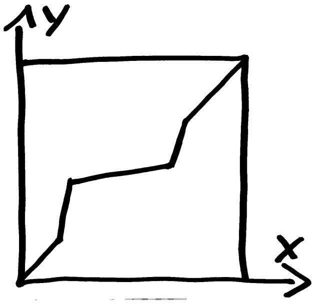
and finally

$$x_1 \cong x_2 = x_0^{-1} x_1 x_0 \mod N.$$

Hence $x_0 x_1 \cong x_1 x_0 \mod N$ which implies that the factor F/N is abelian since x_0 and x_1 generate F. **q.e.d.**

Proof of Theorem (4.34). As we have already seen (4.35), we only have to prove that the commutator subgroup $[F,F]$ is simple. So let $N \trianglelefteq [F,F]$ be a non-trivial normal subgroup and let n be a non-trivial element in N.

Working in the unit interval model for $F = F(\frac{1}{2}, \mathbb{I})$, we draw n:

$n =$ 

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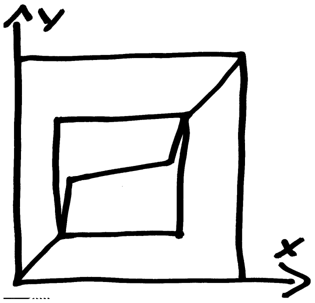
Let $J \subset \mathbb{I}$ be the support of n. Then the picture becomes this:

$n =$ 

.

Note that $F(\frac{1}{2}, J)$ is an isomorphic copy of $F(\frac{1}{2}, \mathbb{I})$ inside $\left[F(\frac{1}{2}, \mathbb{I}), F(\frac{1}{2}, \mathbb{I})\right]$. Moreover, n satisfies the hypothesis of (4.40) inside $F(\frac{1}{2}, J)$. Hence

$$\left[F(\frac{1}{2}, J), F(\frac{1}{2}, J)\right] \leq N \cap F(\frac{1}{2}, J).$$

On the other hand, any element of $\left[F(\frac{1}{2}, \mathbb{I}), F(\frac{1}{2}, \mathbb{I})\right]$ is conjugate inside $\left[F(\frac{1}{2}, \mathbb{I}), F(\frac{1}{2}, \mathbb{I})\right]$ to an element of $\left[F(\frac{1}{2}, J), F(\frac{1}{2}, J)\right]$. Hence the normal closure of $\left[F(\frac{1}{2}, J), F(\frac{1}{2}, J)\right]$ in $\left[F(\frac{1}{2}, \mathbb{I}), F(\frac{1}{2}, \mathbb{I})\right]$ is the whole commutator subgroup $\left[F(\frac{1}{2}, \mathbb{I}), F(\frac{1}{2}, \mathbb{I})\right]$. Hence $\left[F(\frac{1}{2}, \mathbb{I}), F(\frac{1}{2}, \mathbb{I})\right] = N$. **q.e.d.**

Corollary 4.41. *Thompson's group F is not residually finite.*

Corollary 4.42. *Since F is infinite, any finite quotient is proper whence abelian. Therefore, the elements of the commutator subgroup cannot be separated from the identity in any finite quotient. The commutator subgroup, however, is non-trivial – in fact, it is infinite.*