Remark 2.36. As a matter of fact, a group has a paradoxical decomposition if and only if it is not amenable. As a criterion to check this, however, using flows or a paradoxical partition of unity is more easy.

2.4 Stallings' Theorem

The goal of this section is the following characterization of free groups

Theorem 2.37 (Stallings [Stal68]). *A finitely generated group G is free if and only if it has cohomological dimension 1.*

The cohomological dimension $\operatorname{cd} G$ of a group G is an element of $\mathbb{N} \cup \{\infty\}$, and we will define this number in (2.4.1). But we shall outline the proof right away to motivate the exposition. We will argue by induction on the size of a minimal generating set.

Definition 2.38. The <u>rank</u> of a finitely generated group G is the size $\operatorname{rk}(G)$ of a generating set of minimal size.

Exercise 2.39. Prove that $\operatorname{rk}(F_n) = n$.

We need the following facts:

1. [2.57] $\operatorname{cd} G = 0$ if and only if $G = 1$.
2. [2.56] If $\operatorname{cd} G < \infty$, then G is torsion free.
3. [2.55] If $H \leq G$, then $\operatorname{cd} H \leq \operatorname{cd} G$.
4. [2.58] If $\operatorname{cd} G = 1$, then $\operatorname{e}(G) \geq 2$.
5. [2.40] If $\operatorname{e}(G) = 2$ and G is torsion free, the G is the infinite cyclic group.
6. [2.62] If $\operatorname{e}(G) = \infty$ and G is torsion free, then $G = A * B$ for two non-trivial subgroups A and B.

7. For finitely generated groups A and B,

$$\operatorname{rk}(A * B) = \operatorname{rk}(A) + \operatorname{rk}(B)$$

This is known as Grushko's Theorem. [Grus40]

Proof of Stallings' Theorem. First observe that the group G is torsion free because of (2). We induct on $\operatorname{rk}(G)$. If $\operatorname{rk}(G) = 1$, we have a cyclic group which must be C_∞ since this is the only torsion free cyclic group.

So assume $\operatorname{rk}(G) > 1$. From (4) we know that G has at least two ends. If it had two ends, it would be virtually cyclic by (1.61). Since G is torsion free, by (5), it had to be C_∞ which has rank 1. Hence G has infinitely many ends. Now, (6) implies that

$$G = A * B$$

for some finitely generated, non-trivial subgroups A and B.

By (3), $\operatorname{cd} A \leq 1$, and by (1), $\operatorname{cd} A = 1$. Finally Grushko's Theorem implies $\operatorname{rk}(A) < \operatorname{rk}(G)$ whence we can infer by induction that A is a free group.

Analogously, B is free. Hence G is a free product of free groups and, therefore, free. **q.e.d.**

Exercise 2.40. Show that a torsion free group that contains an infinite cyclic subgroup of finite index is infinite cyclic. Remark: There is no hint for this problem because I want you to find a short, elegant (and probably new) solution.

2.4.1 Cohomology of Groups and the Eilenberg-Ganea Problem

Theorem and Definition 2.41. Let G be a group, R a commutative ring with unity $1 \neq 0$, and M a (left) RG-module – note that the involution $g \mapsto g^{-1}$ allows us to regard any left RG-module as a right module and vice versa. For any projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to R \to 0$$

the homology groups

$$\mathrm{H}_*(G;M) := \mathrm{H}_*(P_* \otimes_{RG} M)$$

and cohomology groups

$$\mathrm{H}^*(G;M) := \mathrm{H}^*(\mathrm{Hom}_{RG}(P_*, M))$$

are independent of the chosen projective resolution.

It is, of course, crucial to find nice resolutions.

Example 2.42. For the finite cyclic group, we can cook up a very nice periodic resolution. Observe that

$$RC_n = R[t] / \langle t^n \rangle .$$

With this identification, we can write down the following resolution:

$$\cdots \xrightarrow{\times\left(t^{n-1}+\cdots+1\right)} RG \xrightarrow{\times(t-1)} RG \xrightarrow{\times\left(t^{n-1}+\cdots+1\right)} RG \xrightarrow{\times(t-1)} RG \xrightarrow{\varepsilon} R \to 0$$

From this resolution, we get:

$$\begin{aligned} \mathrm{H}_i(C_n;\mathbb{Z}) &= \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}_n & i \text{ is odd} \\ 0 & i \text{ is even and } > 0 \end{cases} \\ \mathrm{H}^i(C_n;\mathbb{Z}) &= \begin{cases} \mathbb{Z}_n & i \text{ is even} \\ 0 & i \text{ is odd} \end{cases} \end{aligned}$$

Example 2.43. Let X be a contractible simplicial complex upon which G acts freely. That is, no simplex is fixed by any group element. Then the action induces an action of G on the simplicial chain complex $\mathrm{C}_*(X;R)$ which thereby turns into a chain complex of RG-modules. Theses modules are free since G acts freely on X. Since X is contractible, the simplicial chain complex is exact.

Appending the augmentation map (sum up the coefficients on vertices) yields a free resolution

$$\cdots \mathrm{C}_2(X;R) \xrightarrow{\partial} \mathrm{C}_1(X;R) \xrightarrow{\partial} \mathrm{C}_0(X;R) \xrightarrow{\varepsilon} R$$

which can be used to compute the (co)homology of G.

How to come up with a good candidate for X? A generic choice would be the infinite join

$$X := \mathop{\ast}_{\mathbb{N}} G$$

with the diagonal group action. In particular, we can always find a resolution of the trivial RG-module R by free modules.

Example 2.44. Instead of a simplicial complex, one could use a cell complex provided the group acts freely on cells. One way to construct a contractible, free G-complex is to start with a Cayley graph for G. Glue in free G-sets of 2-cells to kill loops. This might introduce non-trivial π_2. Glue in free G-sets of 3-cells to get rid of this, and continue to kill all fundamental groups.

The result is a contractible CW-complex with a free G-action. The advantage is that the 1-skeleton still looks like the Cayley graph.

Definition 2.45. The <u>cohomological dimension over R</u> of a group G is the least element $\operatorname{cd} G \in \mathbb{N} \cup \{\infty\}$ satisfying

$$\mathrm{H}^i(G;-) = 0 \text{ for all } i > \operatorname{cd}_R G.$$

Definition 2.46. The <u>geometric dimension</u> of a group G is the smallest dimension $\operatorname{gd} G$ of a contractible simplicial complex upon which G can act freely.

Observation 2.47. *Using the resolution of (2.43), we see*

$$\operatorname{cd}_R G \leq \operatorname{gd} G$$

Observation 2.48. *For $H \leq G$, we have* $\operatorname{gd} H \leq \operatorname{gd} G$.

Observation 2.49. *Only the trivial group has geometric dimension 0. A group is free if and only if it has geometric dimension 1.*

Hence, we can restate Stallings' Theorem as follows.

Theorem 2.50. *Any finitely generated group G of cohomological dimension 1 has geometric dimension 1.*

This is more in line with the following

Theorem 2.51 (Eilenberg-Ganea [EiGa57]). *For any group G,*

$$\operatorname{gd} G \leq \max(\operatorname{cd} G, 3).$$

Remark 2.52. Settling the case $\operatorname{cd} G = 2$ is the Eilenberg-Ganea Problem. It is generally believed that there are groups of cohomological dimension 2 with geometric dimension 3. In fact, some particular groups are conjectured to have this property. However, as of now, all methods of estimating the geometric dimension of groups are based on homological machinery. Hence, we do not have a prove that one of the alleged examples actually does the trick.

Fact 2.53. *If $m = \operatorname{cd}_R G < \infty$, and*

$$P_{m-1} \to \cdots \to P_1 \to P_0 \to R \to 0$$

is a partial projective resolution, then the kernel

$$P_m := \ker(P_{m-1} \to P_{m-2})$$

is projective. In particular, there is a finite projective resolution

$$0 \to P_m \to \cdots \to P_1 \to P_0 \to R \to 0$$

for R.

Observation 2.54. *Let H be a subgroup of G. Any free RG-module is a free RH-module. Hence any projective RG-module, begin a direct summand of a free RG-module, is a fortiori a projective RH-module.*

Corollary 2.55. *For $H \leq G$, we have $\mathrm{cd}_R H \leq \mathrm{cd}_R G$.* **q.e.d.**

Since a group with torsion has a finite cyclic subgroup, we can infer from (2.42):

Corollary 2.56. *If $\mathrm{cd}\, G < \infty$, the group G is torsion free.* **q.e.d.**

Proposition 2.57. *If $\mathrm{cd}\, G = 0$, then G is trivial.*

Proof. By (2.53), $\mathbb{Z}$ is a projective $\mathbb{Z}G$-module. Hence the augmentation map $\mathbb{Z}G \to \mathbb{Z}$ splits. The image of $1 \in \mathbb{Z}$ under the split must be a G-invariant non-trivial element of $\mathbb{Z}G$. Hence it has constant coefficients for all group elements. This can only happen for finite G as only finitely many coefficients can be non-zero. Hence G is finite. On the other hand, G is torsion free. **q.e.d.**

Exercise 2.58. Let G be finitely generated. Show that $\mathrm{cd}\, G = 1$ implies that G has at least two ends. Hint: Let Γ be a Cayley graph for G. Relate several cohomology theories of Γ and G with coefficients in $\mathbb{Z}_2$ and $\mathbb{Z}_2 G$.

2.4.2 Groups acting freely on trees

We want to detect a non-trivial splitting of a group as a free product. For this purpose the following geometric criterion comes in handy.

Proposition 2.59. *Let G be a group acting on a tree T without terminal vertices and at least some branch points. Assume that the induced action on the set of geometric edges is free and transitive. Then G splits non-trivially as a free product.*

Proof. Since the group acts transitively on the edges, there are either two orbits of vertices or the action on the vertices is transitive, too. The first case is dealt with in (2.61) and the second case is done in (2.60). **q.e.d.**

Lemma 2.60. *Let G act on a tree T such that the following conditions are satisfied:*

1. *G acts freely and transitively on the set of geometric (unoriented) edges.*

2. *G does act transitively on the set of vertices.*

3. *T has no terminal vertices and is not isomorphic to a line.*

Let e be an edge in T that connects the vertices v and w. Let G_v denote the stabilizer of v and let g be an element in G such that $gv = w$. Then $G = G_v * \langle g \rangle$.

Proof. Let G_w denote the stabilizer of w. Let U be an open neighborhood of e. The tree T is connected and covered by the G translates of U. Hence G is generated by

$$\{g \in G \mid gU \cap U \neq \emptyset\} = G_v \cup \{g\} \cup G_w$$

On the other hand $G_w = gG_vg^{-1}$. Hence

$$G = \langle G_v, g \rangle$$

Note that g has infinite order and shifts e to a neighboring edge. Hence we can construct a bi-infinite geodesic C upon which g acts as a unit shift.

Now, consider the action of G on $\partial_\infty T$. We define two subsets. Let $\mathcal{E}_v$ be the set of ends represented by geodesic paths starting at v avoiding C, and let $\mathcal{E}_e$ be the complement of $\mathcal{E}_v$. Obviously, every non-trivial power of g moves $\mathcal{E}_v$ into $\mathcal{E}_e$.

On the other hand, a non-trivial element of G_v cannot move e to $g^{-1}e$ since otherwise G would flip the orientation of an edge and therefore act with non-trivial edge stabilizers. Hence non-trivial elements of G_v take $\mathcal{E}_e$ into $\mathcal{E}_v$.

Since G acts transitively on the set of edges and every vertex has degree ≥ 3, the stabilizer G_v is non-trivial. On the other hand $\langle g \rangle$ is infinite cyclic. Thus, the Ping Pong Lemma (2.14) applies. Hence $G = \langle G_v, g \rangle = G_v * \langle g \rangle$. **q.e.d.**

Exercise 2.61. Let G act on a tree T such that the following conditions are satisfied:

1. G acts freely and transitively on the set of geometric (unoriented) edges.
2. G does not act transitively on the set of vertices.
3. T has no terminal vertices and is not isomorphic to a line.

Let e be an edge in T that connects the vertices v and w. Let G_v and G_w denote the stabilizers of these vertices. Then $G = G_v * G_w$.

2.4.3 Building Trees

This section is devoted to the proof of the following

Theorem 2.62. *Let G be a finitely generated torsion free group with infinitely many ends. Then $G = A * B$ for some non-trivial subgroups A and B.*

The idea of the proof is to replace a Cayley graph for G by a tree that somewhat interpolates between the ends of the Cayley graph. The group G will act on this tree, and then we can use (2.59) to ensure a splitting. The argument is based on the proof in [DiDu89].

So our goal is to find a nice tree for G to act on. We will actually find the set of edges first (they correspond to splittings of the Cayley graph) – more precisely, we will find the oriented edges of the tree. Then we will have to make up the vertices. This process is completely formal and motivates the definition of tree sets.

Definition 2.63. A tree set is a set T together with a fixpoint free involution $\overline{(-)}: T \to T$ and a binary relation $\to\!\!\!\!\!\!-$ satisfying the following axioms:

(ordering) The relation $\to\!\!\!\!\!\!-$ is a partial ordering.

(involution) $t \to\!\!\!\!\!\!- t' \Longleftrightarrow \overline{t'} \to\!\!\!\!\!\!- \bar{t}$.

(nestedness) For any two elements $t, t' \in T$ exactly one of the following six cases occurs:

$$t = t', \;\; \bar{t} = t', \;\; t \to\!\!\!\!\!\!- t', \;\; \bar{t} \to\!\!\!\!\!\!- t', \;\; t \to\!\!\!\!\!\!- \overline{t'}, \;\; \bar{t} \to\!\!\!\!\!\!- \overline{t'}.$$

(finiteness) For any element $t \in T$, the set $T_t := \{t' \in T \mid t' \to\!\!\!\!\!\!- t\}$ contains no infinite chain $t'_1 \to\!\!\!\!\!\!- t'_2 \to\!\!\!\!\!\!- t'_3 \to\!\!\!\!\!\!- \cdots$.

Remark 2.64. It follows from the axioms (2.63)–(2.63) alone that, for any two element $t_1, t_2 \in T$, the interval

$$[t_1, t_2] := \{t' \in T \mid t_1 \to\!\!\!\!\!\!- t' \to\!\!\!\!\!\!- t_2\}$$

is totally ordered. Hence axiom (2.63) just states that intervals are finite.

Exercise 2.65. Show that intervals in tree sets are totally ordered.

Exercise 2.66. Given a tree set T, we construct a graph as follows: the set of oriented edges is T, the vertex set is $\mathcal{V} := T/\sim$ where

$$t \sim t' :\Longleftrightarrow t \to\!\!\!\!\!\!- \overline{t'} \text{ and } \left[t, \overline{t'}\right] = \emptyset,$$

the endpoint map $\tau : T \to \mathcal{V}$ is the canonical projection $T \to T/\sim$, and the initial vertex map $\iota : T \to \mathcal{V}$ is given by $\iota(t) := \tau\left(\bar{t}\right)$. Show that this graph is a tree – here we use the convention that two opposite oriented edges form one geometric edge, thus avoiding bigons.

Lemma 2.67. *Let Γ be a Cayley graph for G with respect to a finite generating system. If Γ has more than three ends, any infinite connected subgraph Δ with finite boundary comprises at least two ends of Γ.*

Proof. Let C be a finite connected subgraph containing $\partial(\Delta)$ with at least three infinite complementary components. Chose a translate gC inside Δ – this exists as Δ is infinite: pick a translate as far away as to make sure it does not intersect C nor the finite complementary regions (of which there are only finitely many as C has finite boundary).

As the Cayley graph is homogeneous, gC will split the space of ends into at least three non-empty subsets. As $\partial(\Delta) \subseteq C$ and C is connected, at least two of these are covered by Δ. **q.e.d.**

Proposition 2.68. *Let G be a finitely generated group with infinitely many ends. Then there is a tree set T upon which G acts with finite stabilizers and at most two orbits.*

Proof. Let Γ be a Cayley graph for G over a finite generating set, and define

$$\mathcal{H}_i := \left\{ U \subset \Gamma \,\middle|\, |U| = \infty = \left|\overline{U}\right| \text{ and } |\partial(U)| < i \right\}.$$

Since Γ has more than one end, these sets are non-empty for sufficiently large i. Let m be the least index for which $\mathcal{H} := \mathcal{H}_m \neq \emptyset$. Note that, because of minimality, the vertex collections in $\mathcal{H}$ are "connected", i.e., for each $U \in \mathcal{H}$, any two points in U can be joined by a path that passes through vertices in U only.

Claim A. Every infinite descending chain $U_1 \supsetneq U_2 \supsetneq \cdots$ in $\mathcal{H}$ has empty intersection.

Proof. Assume the chain had non-empty intersection U_∞. Pick an index i_1. As U_{i_1} is connected, there is an edge e_{i_1} connecting U_∞ to $U_{i_1} - U_\infty$. Hence there is an index i_2 such that, for any $j \geq i_2$, the edge e actually bridges between U_j and $U_{i_1} - U_j$. In particular, $e \in \partial(U_j)$.

Now replace i_1 by i_2 and argue in exactly the same way, to find a new edge e_{i_2} and an index i_3 as above. Observe that $e_{i_2} \neq e_{i_1}$.

Once you constructed $e_{i_{m+1}}$, observe that $\partial\left(U_{i_{m+2}}\right)$ has more than m edges. This contradicts or definition of $\mathcal{H}$. □

Let $U_0 \in \mathcal{H}$ be minimal with $1 \in U_0$.

Claim B. For any group element $V \in \mathcal{H}$, at least one of the following set is finite:

$$U_0 \cap V, \ \ \overline{U_0} \cap V, \ \ U_0 \cap \overline{V}, \ \ \overline{U_0} \cap \overline{V}. \tag{2}$$

PROOF. Assume by contradiction, all of these intersections are infinite. Check that

$$4m \le \partial\left(U_0 \cap V\right) + \partial\left(\overline{U_0} \cap V\right) + \partial\left(U_0 \cap \overline{V}\right) + \partial\left(\overline{U_0} \cap \overline{V}\right) \le 4m$$

Consider the intersection that contains 1, say $U_0 \cap \overline{V}$. Obviously, $1 \in U_0 \cap \overline{V} \subseteq U_0$. But on the other hand, $U_0 \cap \overline{V} \in \mathcal{H}$. By minimality of U_0, we have $U_0 \cap \overline{V} = U_0$, whence $V \cap U_0 = \emptyset$. Thus, not all intersections are infinite. □

Claim C. If two of the intersections in (2) are finite, it is either the pair

$$U_0 \cap V \quad \overline{U_0} \cap \overline{V}$$

or the pair

$$\overline{U_0} \cap V \quad U_0 \cap \overline{V}.$$

PROOF. Consider the diagram

$$\begin{array}{ccccc} (U_0 \cap V) & \cup & \left(U_0 \cap \overline{V}\right) & = & U_0 \\ \cup & & \cup & & \\ \left(\overline{U_0} \cap V\right) & \cup & \left(\overline{U_0} \cap \overline{V}\right) & = & \overline{U_0} \\ \shortparallel & & \shortparallel & & \\ V & & \overline{V} & & \end{array}$$

Since U_0, $\overline{U_0}$, V, and $\overline{V}$ are all infinite, the claim follows. □

Claim D. Let $U, V \in \mathcal{H}$ such that $\overline{U} \cap U_0$ is finite, then there are only finitely many $g \in G$ such that

$$\overline{U} \cap V \text{ and } \overline{V} \cap U_0 \text{ are finite.}$$

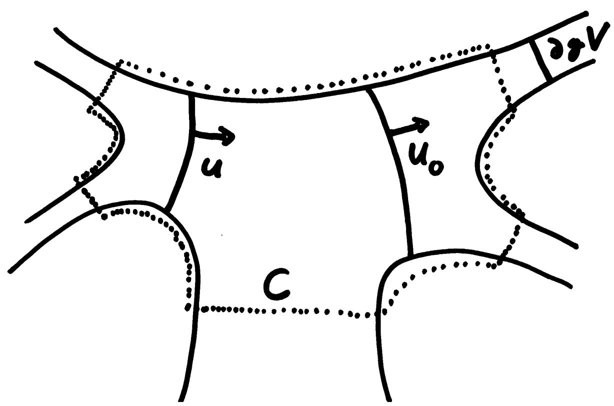
Proof. Because of (2.67), there is a compact subset C in the Cayley graph Γ of G such that the intersections

$$\overline{U} \cap \overline{C}$$

and

$$\overline{U_0} \cap \overline{C}$$

contain at least two infinite components.



Choose a finite connected subgraph Δ containing $\partial(V)$. Then for all but finitely many $g \in G$, the translate $g\partial(V) \subseteq g\Delta$ is contained in an infinite complementary component of C. For these g, it is impossible that $\overline{U} \cap V$ and $\overline{V} \cap U_0$ are finite. □

We define an equivalence relation on $\mathcal{H}$ by

$$U \cong V :\Longleftrightarrow \overline{U} \cap V \text{ and } U \cap \overline{V} \text{ are finite.}$$

In addition, we define a partial order relation $\rightarrow$ on $\mathcal{H}$ by

$$U \rightarrow V :\Longleftrightarrow \overline{U} \cap V \text{ is finite but } U \ncong V.$$

Put

$$T := \{gU\} \cup \{g\overline{U}\} \big/ _{\cong}$$

where

$$U \cong V :\Longleftrightarrow (U \rightarrowtail V \text{ and } V \rightarrowtail U)\,.$$

Obviously $\rightarrowtail$ descends and defines a partial order on T which we will also denote by $\rightarrowtail$.

Claim E. T is a tree set.

Proof. Obviously, $\rightarrowtail$ is a partial order, and clearly taking complements is an order reversing involution. The condition (2.63) is satisfied by claim (B). Finally, intervals are finite by (D). **q.e.d.**

So finally, we have constructed the tree set. What about the action of G? The number of orbits is clearly bounded by two since we used the orbits of U_0 and $\overline{U_0}$ to define T. That the stabilizers are finite, follows from (D) for the case $U = U_0$. **q.e.d.**

Finally we finish this section with the

Proof of (2.62). By (2.68) there is a tree set upon which G acts with finite stabilizers and at most two orbits. Since G is torsion free, the stabilizers are actually trivial. Hence the action on the corresponding tree is free on oriented edges and transitive on geometric edges since there are at most two orbits of oriented edges.

Suppose the stabilizer of a geometric edge was non-trivial. Then the squares of its elements would be trivial as they stabilize an oriented edge. Since the group is torsion free, this cannot happen. **q.e.d.**