1 The infinite cyclic group

The infinite cyclic group $C_\infty = \mathbb{Z}$ is the most simple infinite discrete group. It can be given in various ways:

- by a presentation $C_\infty = \langle x \rangle$.
- as the fundamental group of the circle $C_\infty = \pi_1(\mathbb{S}^1)$.
- as a group of matrices $C_\infty = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \middle| \, s \in \mathbb{Z} \right\}$.

So we can read off that C_∞ is finitely presented and linear. It acts freely and cocompactly on the real line, the universal cover of $\mathbb{S}^1$. The real line is contractible and has dimension 1. Hence C_∞ is of type F. What else can we say about C_∞?

1.1 Residual Finiteness

Definition 1.1. A group G is <u>residually blah</u> if for every non-trivial element g, there is a blah quotient of G wherein g does not become trivial.

Theorem 1.2. *C_∞ is residually finite.*

Proof. The infinite cyclic group has factors of any order. An element will not be trivial in any factor whose order is relatively prime to the order of the given element. **q.e.d.**

1.2 Amenability

Definition 1.3. Let G be a group acting on a set M. The set M is called <u>G-amenable</u> if there is a G-invariant finitely additive probability measure on the system of all subsets of M.

A group G is called <u>amenable</u> if it is a G-amenable set with respect to left multiplication.

Example 1.4. The counting measure shows that finite groups are amenable.

Remark 1.5. Such a measure allows to do the usual averaging trick in representation theory.

For groups, amenability can be strengthened.

Proposition 1.6. *An amenable group G also has a bi-invariant finitely additive probability measure $\tilde{\mu}$.*

Proof. Let μ be a left-invariant finitely additive probability measure on G. Observe that

$$\mu^{-}(S) := \mu\left(S^{-1}\right)$$

defines a right-invariant measure on G. It is easy to check that

$$\tilde{\mu}(S) := \int_{g \in G} \mu\left(Sg^{-1}\right) \mathrm{d}_{\mu^{-}}\, g$$

is bi-invariant: We have

$$\begin{aligned} \tilde{\mu}(hS) &= \int_{g \in G} \mu\left(hSg^{-1}\right) \mathrm{d}_{\mu^{-}}\, g \\ &= \int_{g \in G} \mu\left(Sg^{-1}\right) \mathrm{d}_{\mu^{-}}\, g \tilde{\mu}(S) \end{aligned}$$

and

$$\begin{aligned} \tilde{\mu}(Sh) &= \int_{g \in G} \mu\left(Shg^{-1}\right) \mathrm{d}_{\mu^{-}}\, g \\ &= \int_{g \in G} \mu\left(Shg^{-1}\right) \mathrm{d}_{\mu^{-}}\, gh^{-1} \\ &= \int_{g \in G} \mu\left(S\left(gh^{-1}\right)^{-1}\right) \mathrm{d}_{\mu^{-}}\, gh^{-1} \\ &= \tilde{\mu}(S) \end{aligned}$$

q.e.d.

In this section we show

Theorem 1.7. *The infinite cyclic group is amenable.*

1.2.1 Følner Sequences

Definition 1.8. A Følner sequence for a G-set M is a sequence (F_i) of finite subsets $F_i \subseteq M$ such that

$$\lim_{i\to\infty} \frac{|gF_i \triangle F_i|}{|F_i|} = 0 \qquad \text{for all } g \in G$$

Here $M \triangle N$ denotes the symmetric difference of two sets:

$$M \triangle N := (M \cup N) - (M \cap N)$$

Exercise 1.9. *For any sequence (F_i) of finite subsets in M, the set*

$$\left\{ g \in G \,\middle|\, \lim_{i\to\infty} \frac{|gF_i \triangle F_i|}{|F_i|} = 0 \right\}$$

is a subgroup of G.

Lemma 1.10. *The "balls of radius i" form a Følner sequence for the infinite cyclic group.*

Proof. The ball of radius i is the subset $\{-i, -i+1, \ldots, i-1, i\}$. It is obvious that for a fixed group element g, very large balls will have large overlaps with their g-translate. **q.e.d.**

We will show below, that a group is amenable if it has a Følner sequence. This will complete the proof that the infinite cyclic group is amenable.

1.2.2 Ultralimits

Definition 1.11. Let M be a set. A filter on M is a set $\mathcal{F}$ of subsets of M satisfying:

1. $\emptyset \notin \mathcal{F}$.
2. If $F \in \mathcal{F}$ and $M \supseteq H \supseteq F$, then $H \in \mathcal{F}$.
3. If $F \in \mathcal{F}$ and $H \in \mathcal{F}$, then $(F \cap H) \in \mathcal{F}$.

Observation 1.12. *Since a filter does not contain the empty set but is closed with respect to forming intersections, no two sets in a filter are disjoint.* **q.e.d.**

Example 1.13. For any non-empty subset $S \subseteq M$, the system

$$\{F \subseteq M \mid F \supseteq S\}$$

is a filter. It is called the principal filter induced by S.

Example 1.14. If M is infinite, the set of cofinite sets (complements of finite sets) is a filter $\mathcal{CF}_M$.

Example 1.15. A directed set is a partially ordered set D such that any two elements have a common upper bound. A coinitial segment is a subset $F \subseteq D$ satisfying:

if $\alpha \in F$ and $\alpha < \beta$, then $\beta \in F$.

The set of supersets of non-empty coinitial segments is a filter $\mathcal{CI}_T$.

Exercise 1.16. *Show that* $\mathcal{CF}_{\mathbb{N}} = \mathcal{CI}_{\mathbb{N}}$.

All filters on M are comparable with respect to inclusion: a filter $\mathcal{F}$ is called finer that a filter $\mathcal{H}$ if every subset of M that belongs to $\mathcal{H}$ also belongs to $\mathcal{F}$. Obviously, this just means $\mathcal{H} \subseteq \mathcal{F}$. A filter that cannot be refined (a finest filter) is called an ultrafilter.

Example 1.17. Fix a subset $S \subset M$ and a filter $\mathcal{F}$ that contains neither S nor $M - S$. Then

$$\mathcal{F}_S := \{T \mid M \supseteq T \supseteq (S \cap F) \text{ for some } F \in \mathcal{F}\}$$

is a filter finer than $\mathcal{F}$.

From this example, we immediately infer:

Lemma 1.18. *Let $\mathcal{U}$ be an ultrafilter on M. For every subset $S \subseteq M$ either $S \in \mathcal{U}$ or $(M - S) \in \mathcal{U}$.* **q.e.d.**

Since ascending unions of filters are filters, Zorn's Lemma immediately implies:

Lemma 1.19. *Every filter is contained in an ultrafilter.* **q.e.d.**

Definition 1.20. Let M be a set, $\mathcal{F}$ a filter on M, and X a topological space. A family $(x_m)_{m \in M}$ of points in X $\mathcal{F}$-converges to a point $x \in X$ if, for every open neighbourhood U of x,

$$\{m \in M \mid x_m \in U\} \in \mathcal{F}.$$

In this case, we say $x = \mathcal{F}\text{-}\lim_{m \in M} x_m$ is an $\mathcal{F}$-limit of n.

Example 1.21. Ordinary convergence of sequences is the same as $\mathcal{CF}_{\mathbb{N}}$-convergence.

Example 1.22. A net in X over D is a family of points in X indexed by a directed set D. Convergence for nets is defined as $\mathcal{CI}_D$-convergence.

Observation 1.23. *If $\mathcal{H}$ is finer than $\mathcal{F}$ then any $\mathcal{F}$-limit of a net is also an $\mathcal{H}$-limit.*

Proposition 1.24. *If X is Hausdorff, then $\mathcal{F}$-limits are unique.*

Proof. This is done by contradiction. Suppose there were two points a_1 and a_2 such that, for each open neighbourhood U_j of a_j,

$$\{m \in M \mid x_m \in U_j\} \in \mathcal{F}.$$

By Hausdorffness, we can choose these two neighbourhoods to be disjoint. But then, both sets

$$\{m \in M \mid x_m \in U_j\}$$

are disjoint. As $\mathcal{F}$ is a filter, this cannot happen. **q.e.d.**

Theorem 1.25. *Fix an ultrafilter $\mathcal{U}$ on the set M. Let C be a compact topological space. Then any family of points $(x_m)_{m\in M}$ has a $\mathcal{U}$-limit.*

Proof. Suppose that each $a \in C$ has an open neighbourhood U such that

$$\{m \in M \mid x_m \in U\} \notin \mathcal{U}$$

then we can, by compactness, cover C with finitely many open sets U_j such that for each j

$$\{m \in M \mid x_m \in U_j\} \notin \mathcal{U}.$$

However,

$$M = \bigcup_j \{m \in M \mid x_m \in U_j\}$$

is a cover of M by finitely many subsets. Hence one of them must be in $\mathcal{U}$ by the following Lemma 1.26 **q.e.d.**

Lemma 1.26. *Let $\mathcal{U}$ be an ultrafilter. Suppose a finite union $S_1 \cup \cdots S_n$ belongs to $\mathcal{U}$, then one of the summands is in $\mathcal{U}$, i.e., for at least one index i, $S_i \in \mathcal{U}$.*

Proof. Suppose $S_i \notin \mathcal{U}$ for all i. Then

$$M - S_i \in \mathcal{U} \qquad \text{for all } i$$

Hence

$$\bigcap_i M - S_i = M - \bigcup_i S_i \in \mathcal{U}$$

which contradicts the assumption $\bigcup_i S_i \in \mathcal{U}$. **q.e.d.**

Lemma 1.27. *Let $f : X \to Y$ be a continuous map between compact spaces. Then for any $\mathcal{F}$-convergent family $(x_m)_{m\in M}$ in X,*

$$\mathcal{F}\text{-}\lim f(x_m) = f\left(\mathcal{F}\text{-}\lim x_m\right).$$

Proof. Consider an open neighbourhood U of $f(\mathcal{F}\text{-}\lim x_m)$. Its preimage under f is an open neighbourhood V of $\mathcal{U}\text{-}\lim x_m$. Hence

$$\{m \in M \mid f(x_m) \in U\} = \{m \in M \mid x_m \in V\} \in \mathcal{U}.$$

Since U was arbitrary, the statement follows. **q.e.d.**

Remark 1.28. Let C and D be compact. For any family of pairs $((x_\alpha, y_\alpha))$ in $C \times D$, we have:

$$\mathcal{U}\text{-}\lim(x_\alpha, y_\alpha) = (\mathcal{U}\text{-}\lim x_\alpha, \mathcal{U}\text{-}\lim y_\alpha)$$

Corollary 1.29. *Bounded sequences of real numbers have unique ultralimits and these limits are compatible with the arithmetic operations of addition, subtraction and multiplication.*

1.2.3 From Følner Sequences to Amenability

Proposition 1.30. *A G-set M is G-amenable if it admits a Følner sequence.*

Proof. Suppose we have a Følner sequence (F_i). Fix an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ refining the coinitial filter so that we can form ultralimits of bounded sequences of real numbers. Let S be any subset of M. Then

$$\frac{|S \cap F_i|}{|F_i|}$$

is a sequence in $[0,1]$. We define the probability measure μ by:

$$\mu(S) := \mathcal{U}\text{-}\lim \frac{|S \cap F_i|}{|F_i|}$$

Obviously $\mu(G) = 1$.

To see that this measure is additive, consider two disjoint subsets S and T and observe that

$$|(S \cup T) \cap F_i| = |S \cap F_i| + |T \cap F_i|$$

From this additivity follows because ultralimits commute with addition (1.29).

To see that the measure is left-invariant, we write

$$\left|\frac{|gS\cap F_i|}{|F_i|}-\frac{|S\cap F_i|}{|F_i|}\right|=\left|\frac{|S\cap g^{-1}F_i|}{|F_i|}-\frac{|S\cap F_i|}{|F_i|}\right|\leq\frac{|S\cap(g^{-1}F_i\triangle F_i)|}{|F_i|}\xrightarrow[i\to\infty]{}0$$

Now the claim follows since our ultrafilter refines the cofinite filter on $\mathbb{N}$ whence ordinary limits are ultralimits for $\mathcal{U}$. **q.e.d.**

Corollary 1.31. *The infinite cyclic group is amenable.* **q.e.d.**

Exercise 1.32. *Show that every abelian group is amenable:*

1. *Show that the direct product of two amenable groups is amenable.*
2. *Show that a group is amenable if all its finitely generated subgroups are amenable (i.e., locally amenable groups are amenable). Hint: The system of finitely generated subgroups inside G is a directed set. Use an ultralimit construction to obtain a measure on G from the measures on the finitely generated subgroups of G.*

From (1) infer that finitely generated abelian groups are amenable. Then (2) implies that abelian groups are amenable.

Exercise 1.33. *Add a little twist to what you did on direct products and show that a group is amenable if it has an amenable normal subgroup such that the quotient is also amenable. I.e., amenable-by-amenable groups are amenable. Infer that solvable groups are amenable.*

Exercise 1.34. *Show that subgroups of amenable groups are amenable.*

Lemma 1.35. *If a group G has an amenable subgroup H of finite index, it is amenable.*

Proof. Let K be the kernel of the action of G on the finite set of cosets G/H. Obviously, $K \leq H$. Hence K is amenable. Since we have the short exact sequence

$$K \hookrightarrow G \twoheadrightarrow G/K$$

wherein K is amenable and G/K is finite, G is amenable-by-amenable and therefore amenable. **q.e.d.**

Corollary 1.36. *Virtually solvable groups (i.e., groups that have a solvable subgroup of finite index) are amenable.* **q.e.d.**

1.3 Kazhdan's Property (T)

Definition 1.37. A unitary representation of a topological group G on a Hilbert space $\mathcal{H}$ is said to <u>have almost invariant vectors</u>, if, for any compact subset $K \subseteq G$ and any $\varepsilon > 0$, there is a unit vector $\mathbf{u} \in \mathcal{H}$ satisfying

$$|g\mathbf{u} - \mathbf{u}| < \varepsilon \qquad \text{for all } g \in K.$$

The group G has Property (T) if every unitary representation that has almost invariant vectors has an invariant vector.

Theorem 1.38. *If a discrete group G has a Følner sequence, then G does not have Kazhdan's property (T).*

Proof. Let F_i form a Følner sequence. Consider the action of G on the Hilbert space $L^2(G)$ of square summable function on G. The action is given by a shift. This action has no invariant vectors. But the sequence of vectors

$$\mathbf{u}_i : g \mapsto \begin{cases} \frac{1}{\sqrt{|F_i|}} & g \in F_i \\ 0 & g \notin F_i \end{cases}$$

satisfies

$$\lim_{i\to\infty} |g\mathbf{u}_i - \mathbf{u}_i| = 0 \qquad \text{for all } g \in G$$

From this, the claim follows since compact subsets of discrete groups are finite. **q.e.d.**

Corollary 1.39. *The infinite cyclic group does not have Kazhdan's property (T).* **q.e.d.**

Definition 1.40. A group is indicable if it admits an epimorphism onto the infinite cyclic group.

Observation 1.41. *Every quotient of a Kazhdan (T) group has property (T).* **q.e.d.**

Corollary 1.42. *Indicable groups do not have Kazhdan's property (T).*

Corollary 1.43. *The free groups and the pure braid groups do not have Kazhdan's property (T).*

Lemma 1.44. *A virtually finitely generated group G is finitely generated.*

Proof. Let $H = \langle h_1, \dots, h_r \rangle$ be a finitely generated subgroup of G and let $g_1, \dots, g_s$ be a complete set of representatives for the finitely many cosets in G/H. Then

$$G = \langle H, g_1, \dots, g_s \rangle = \langle h_1, \dots, h_r, g_1, \dots, g_s \rangle .$$

q.e.d.

Proposition 1.45. *A (discrete) group G that has Kazhdan's property (T) is finitely generated.*

Proof. We consider the unitary representation

$$\bigoplus_{H \leq G} L^2(G/H)$$

where H runs through all finitely generated subgroups of G. Since any compact (finite) subset of G is contained in one of these subgroups, this representation has almost invariant vectors – just consider the action of the finite subset on an appropriate summand where it fixes the coset of the identity. Since G is supposed to be Kazhdan, we conclude that the representation has an invariant vector.

Hence one of the summands has an invariant vector. Such a vector corresponds to a constant function on G/H. Hence this quotient is finite. Therefore, G is virtually finitely generated and hence finitely generated. **q.e.d.**

Corollary 1.46. *Locally indicable groups do not have Kazhdan's property (T).*

Proof. Being discrete and Kazhdan, the group is finitely generated. Being finitely generated and locally indicable, it is indicable. **q.e.d.**

Exercise 1.47. *Let G be a finitely generated group. Prove that any subgroup of finite index in G is finiteley generated.*

1.4 The Geometry of the Cayley Graph

Definition 1.48. Let G be a group with finite generating system Σ. The (left) Cayley graph $\Gamma_\Sigma(G)$ is a (directed and labeled) graph. Its set of vertices is G, and for each vertex $g \in G$ and each generator $x \in \Sigma$, there an edge (labeled by x) from g to gx. Note that G acts from the left on $\Gamma_\Sigma(G)$.

There is a corresponding notion of a right Cayley graph upon which G acts from the right.

Remark 1.49. We usually do not care about the direction of edges or the labeling. Thus we regard the Cayley graph as a metric space: every edge has length 1 and the distance of any two points is the