# Math 739 – Important Groups (homework 11, due Apr 24)

**Definition.** The hyperbolic plane is the upper half plane

$$\mathbb{H}^2 = \{x + \mathrm{i}y \in \mathbb{C} \mid y > 0\}$$

with the Riemannian metric

$$\mathrm{d}\, s^2 = \frac{\mathrm{d}\, x^2 + \mathrm{d}\, y^2}{y^2}.$$

**Exercise 11.1.** Show that Möbius transformations are isometries of $\mathbb{H}^2$.

**Exercise 11.2 (extra credit).** Show that any orientation preserving isometry of $\mathbb{H}^2$ is given by a Möbius transformation.

**Exercise 11.3.** Show that geodesics in $\mathbb{H}^2$ are vertical lines or half circles orthogonal to the real axis.

**Definition.** A horizontal line or a circle tangent to the real axis in $\mathbb{H}^2$ is called a horocircle.

**Exercise 11.4.** Show that the action of $\mathrm{SL}_2(\mathbb{R})$ takes horocircles to horocircles.

**Notation.** $\overline{D} := \left\{ z \in \mathbb{H}^2 \;\middle|\; |z| \geq 1 \text{ and } -\frac{1}{2} \leq \Re(z) \leq \frac{1}{2} \right\}.$

**Exercise 11.5.** Show that, for any $M \in \mathrm{SL}_2(\mathbb{Z})$,

$$M\overline{D} \cap \overline{D}$$

does not contain a non-empty open subset of $\mathbb{H}^2$.

**Exercise 11.6.** Let $G$ be a residually finite group that has only finitely many conjugacy classes of finite subgroups. Show that $G$ is virtually torsion free.

**Exercise 11.7.** Show that $\mathrm{SL}_2(\mathbb{Z})$ contains a non-abelian free subgroup of finite index.

It suffices to solve, on the average, half of the problems correctly.