Math 739 – Important Groups (homework 10, due Apr 12)

Exercise 10.1. Show that the polynomial $(a_{21}+1)^3 = 1$ defines a subgroup of $\mathrm{SL}_2(\mathbb{F}_2)$, but it does not define a subgroup of $\mathrm{SL}_2(\mathbb{F}_4)$.

Two subgroups of a common supergroup are called <u>commensurable</u> if their intersection has finite index in both of them.

Exercise 10.2. Let $\varphi : G \to H$ be a $\mathbb{Q}$-isomorphism of $\mathbb{Q}$-groups. Show that $\varphi(G(\mathbb{Z}))$ and $H(\mathbb{Z})$ are commensurable.

Exercise 10.3. Let $\varphi : G \to H$ be a $\mathbb{Q}$-morphism of $\mathbb{Q}$-groups. Show that $\varphi^{-1}(H(\mathbb{Z})) \cap G(\mathbb{Z})$ has finite index in $G(\mathbb{Z})$.

Exercise 10.4. Let k be a finite extension of $\mathbb{Q}$, let $\mathcal{O} \subseteq k$ be the subring of algebraic integers in k, and let G be a k-group. Prove that $G(\mathcal{O})$ is an arithmetic group. Hint: Represent k as a matrix algebra over $\mathbb{Q}$ and use this to find a rational representation of $G(\mathcal{O})$.

Exercise 10.5. Show that any finite group is arithmetic.

Exercise 10.6. Prove that $\mathrm{SL}_n(\mathbb{Z})$ is generated by elementary matrices, i.e., matrices that have 1s in the diagonal and precisely one additional 1 in an off-diagonal slot.

Exercise 10.7 (extra credit). Prove that $\mathrm{SL}_n(\mathbb{Z})$ is generated by two elements for $n \geq 5$. Remark: The statement holds for $n \geq 2$. However, a proof of the more general statement distinguishes between n even and n odd.

Exercise 10.8 (Minkowski (1887)). Show that the kernel of the map

$$\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}_p)$$

is torsion free for any odd prime p. Hint: If M is a torsion element of $\mathrm{SL}_n(\mathbb{Z})$ then the roots of its characteristic polynomial are roots of unity. This tells you something about the way it factors over $\mathbb{Z}$.

It suffices to solve, on the average, half of the problems correctly.