Math 739 – Important Groups (homework 8, due Mar 29)

**Exercise 8.1.** Prove that the map

\[ x_i : t \mapsto \begin{cases} 
  t & \text{for } 0 \leq t \leq i \\
  i + \frac{1}{2}(t - i) & \text{for } i \leq t \leq i + 2 \\
  t - 1 & \text{for } i + 2 \leq t 
\end{cases} \]

defines an action of \( F \) on \([0, \infty]\).

**Exercise 8.2.** Prove that the words with even exponent sum for the generator \( x_0 \) form a subgroup of index 2 in \( F \) which is isomorphic to \( F \).

**Exercise 8.3.** Show that the centralizer of \( x_1 \) in \( F \) is isomorphic to \( F \times C_\infty \).

**Exercise 8.4.** Show that

\[ \langle \ldots x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots \mid x_m x_i = x_i x_{m+1} \text{ for } i < m \rangle \]

embeds into \( F \) as a normal subgroup with infinite cyclic quotient. Note that this group is not finitely generated.

**Definition.** The Baumslag-Solitar group is defined by the presentation

\[ BS(1,2) := \langle x, y \mid x^y = x^2 \rangle \]

**Exercise 8.5.** Show that the presentation complex for \( BS(1,2) \) is contractible and one-ended.

**Exercise 8.6.** Prove that \( BS(1,2) \) has exponential growth.

**Exercise 8.7.** As \( BS(1,2) \) is solvable, we know that it is amenable and has a Følner sequence. Prove that \( BS(1,2) \) has no Følner sequence consisting of balls in the Cayley graph defined by the presentation.

It suffices to solve, on the average, half of the problems correctly.